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The Method of Glowinski and Pironneau for the Unsteady Stokes Problem

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Abstract—The unsteady Stokes problem, i.e., the Stokes problem with a constant multiple of the velocity included in the velocity-pressure equation, is often central to methods used to solve the nonstationary Navier–Stokes equations and the equations governing viscoelastic flows. The Glowinski–Pironneau finite-element method for the Stokes problem decomposes the problem into a series of Poisson’s equations, providing a potentially efficient approach for large problems in two or three dimensions. The goal of this paper is to present a complete development and analysis of the Glowinski–Pironneau method for the unsteady Stokes problem, along with numerical results which confirm the analytical estimates. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Stokes problem, Finite-element method, Glowinski–Pironneau method.

1. INTRODUCTION

The Stokes problem plays a fundamental role in the modeling of incompressible viscous flows. The equations are known to govern slow (low Reynolds number) flows, and perhaps more significantly they are central to the numerical solution of the Navier–Stokes equations [1,2]. The application motivating this work is viscoelastic flow associated with polymeric fiber and film processes. The θ -method is a splitting technique, first developed for the unsteady Navier–Stokes equations [3], and more recently adapted to the equations governing unsteady viscoelastic flows. In the latter case, the nonlinear terms appear in the constitutive equation rather than the momentum equation, and the first and third steps of the three-step θ -method are Stokes solves [4]. In this case, the Stokes problem takes the form

$$\begin{aligned}\eta \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0, & \text{in } \Omega, \\ \mathbf{u}|_{\Gamma} &= \mathbf{u}_b,\end{aligned}\tag{1}$$

with possible variation in the boundary condition. It is assumed that Ω is an open, bounded domain in \mathbb{R}^N , $N = 2$ or 3 , with smooth boundary Γ , $\{\eta, \nu\} \in \mathbb{R}$, $\eta \geq 0$, $\nu > 0$, $\mathbf{f} \in (L^2(\Omega))^n$,

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and $\mathbf{u}_b \in H^{1/2}(\Gamma)$, satisfying

$$\int_{\Gamma} \mathbf{u}_b \cdot \mathbf{n} \, d\Gamma = 0.$$

We shall refer subsequently to the $\eta \neq 0$ case as the *unsteady* Stokes problem [5].

A finite-element solution of a viscoelastic flow problem in two dimensions may involve $O(10^6)$ variables, so for these problems and especially for problems in three dimensions, the development of efficient iterative solvers is essential [6]. The θ -method is gaining acceptance because of its attractive stability properties. The emphasis is then on developing an efficient parallel solver for (1). A promising candidate is the method of Glowinski and Pironneau, which is based on the simple observation that if \mathbf{u} satisfies (1), then

$$\nabla \cdot (\eta \mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla p) = \nabla^2 p = \nabla \cdot \mathbf{f}.$$

If an appropriate boundary pressure $p_{\Gamma} = p|_{\Gamma}$ can be found, then p and \mathbf{u} are solutions of the Poisson problems

$$\begin{aligned} -\nabla^2 p &= -\nabla \cdot \mathbf{f}, & \eta \mathbf{u} - \nu \nabla^2 \mathbf{u} &= \mathbf{f} - \nabla p, \\ p|_{\Gamma} &= p_{\Gamma}, & \mathbf{u}|_{\Gamma} &= \mathbf{u}_b. \end{aligned} \quad (2)$$

The constraint $\nabla \cdot \mathbf{u} = 0$ is used to determine p_{Γ} , indirectly through the unique function θ , satisfying

$$\begin{aligned} \mathbf{u} &= \nabla \theta + \nabla \times \Psi, \\ \theta|_{\Gamma} &= 0. \end{aligned} \quad (3)$$

The Glowinski-Pironneau method is presented for the case $\eta \neq 0$ in [7], for the case $\eta = 0$ in [8] and analyzed in more detail for the case $\eta = 0$ in [9]. Each of these papers refers to a subsequent paper for certain analytical and numerical details. To the best of our knowledge, that paper never appeared, though it is worth noting that the method for the case with $\eta = 0$ is also presented in [1] and [10].

Because the Glowinski-Pironneau algorithm for (1) appears promising as a key component in solving viscoelastic flow problems, the intent of this paper is to present a complete analysis of the method, specifically for the $\eta \neq 0$ case, along with numerical confirmation of convergence estimates for errors in the finite-element approximation. Though this paper is focused on the two-dimensional problem, sufficient generality is included so that the analysis also applies to the problem in three dimensions.

The rest of the paper is outlined as follows. In Section 2, the reformulation of (1), which is fundamental to the Glowinski-Pironneau method, is presented along with necessary regularity properties. The continuous and discrete variational formulations are analyzed in Section 3. Error analysis and computational results are presented in Sections 4 and 5, respectively. In Section 6, conclusions and next steps in this research are discussed.

2. BASIC EQUATIONS AND REGULARITY

As mentioned in Section 1, the potential function θ in the curl-free part of \mathbf{u} plays a role in determining the pressure boundary function, p_{Γ} , and also in imposing the divergence-free constraint on \mathbf{u} . Taking the divergence of both sides of the differential equation in (3) leads to a Poisson problem for θ ,

$$\begin{aligned} -\nabla^2 \theta &= -\nabla \cdot \mathbf{u}, \\ \theta|_{\Gamma} &= 0. \end{aligned} \quad (4)$$

To develop the algorithm for finding p_{Γ} , and the necessary regularity properties of the solution variables, we first consider p , \mathbf{u} , and θ as functions that depend on a prescribed pressure boundary

function g . First decompose p , \mathbf{u} , and θ into g -independent and g -dependent parts, i.e.,

$$p(g) = p_0 + p_1(g), \quad \mathbf{u}(g) = \mathbf{u}_0 + \mathbf{u}_1(g), \quad \theta(g) = \theta_0 + \theta_1(g), \tag{5}$$

$$-\nabla^2 p_0 = -\nabla \cdot \mathbf{f}, \quad \eta \mathbf{u}_0 - \nu \nabla^2 \mathbf{u}_0 = \mathbf{f} - \nabla p_0, \quad -\nabla^2 \theta_0 = -\nabla \cdot \mathbf{u}_0, \tag{6}$$

$$\begin{aligned} p_0|_\Gamma &= 0, & \mathbf{u}_0|_\Gamma &= \mathbf{u}_b, & \theta_0|_\Gamma &= 0, \\ -\nabla^2 p_1(g) &= 0, & \eta \mathbf{u}_1(g) - \nu \nabla^2 \mathbf{u}_1(g) &= -\nabla p_1(g), & -\nabla^2 \theta_1(g) &= -\nabla \cdot \mathbf{u}_1(g), \\ p_1|_\Gamma &= g, & \mathbf{u}_1|_\Gamma &= 0, & \theta_1|_\Gamma &= 0. \end{aligned} \tag{7}$$

Now considering conditions so that weak solutions of (6) and (7) are well defined, first it is assumed that Ω is connected and elliptic regular. That is, if $L(\cdot) = [\eta - \nu \nabla^2](\cdot)$ with $\eta \geq 0, \nu > 0$ and ψ satisfies

$$\begin{aligned} L\psi &= \xi, \\ \psi|_\Gamma &= \gamma, \end{aligned}$$

then

$$\|\psi\|_{k,\Omega} \leq C_1 \left(\|\xi\|_{k-2,\Omega} + \|\gamma\|_{k-1/2,\Gamma} \right) \leq C_2 \|\psi\|_{k,\Omega}, \quad k \in \{1, 2\},$$

and the map $\mu \rightarrow \frac{\partial \mu}{\partial n}$ is continuous and surjective taking $H^2(\Omega) \cap H_0^1(\Omega) \rightarrow H^{1/2}(\Omega)$.

An example of such a domain is a connected bounded open domain of dimension two or three with either a Lipschitz continuous or a convex polyhedral boundary. This condition will be denoted as $\Gamma \in C^*$. Note that in this setting, $H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))'$.

2.1. Regularity

The main regularity properties and two related results are summarized in the following lemma. Though the results are mostly contained in [1], a proof is included here so that parts of the proof may be referenced in a subsequent section.

LEMMA 2.1. Consider equations (5)–(7). If $\Gamma \in C^*$, $\mathbf{f} \in (L^2(\Omega))^N$, and $g \in H^{-1/2}(\Gamma)/\mathfrak{R}$, then

$$p_0 \in H_0^1(\Omega), \quad \mathbf{u}_0 \in (H^2(\Omega))^N, \quad \theta_0 \in H^2(\Omega) \cap H_0^1(\Omega), \tag{8}$$

$$p_1(g) \in L^2(\Omega), \quad \mathbf{u}_1(g) \in (H_0^1(\Omega))^N, \quad \theta_1(g) \in H^2(\Omega) \cap H_0^1(\Omega), \tag{9}$$

and therefore,

$$p(g) \in L^2(\Omega), \quad \mathbf{u}(g) \in (H^1(\Omega))^N, \quad \theta(g) \in H^2(\Omega) \cap H_0^1(\Omega).$$

Also, the linear functional defined by

$$F(g) = - \int_\Gamma \frac{\partial \theta_0}{\partial n} g \, d\Gamma \tag{10}$$

is bounded on $H^{-1/2}(\Gamma)/\mathfrak{R}$, and

$$\|p_1(g)\|_{0,\Omega} \simeq \|g\|_{-1/2,\Gamma}. \tag{11}$$

PROOF. The result (8) is a standard result, following from (6) and $\Gamma \in C^*$ [1]. Thus the inner product in (10) is bounded. To assure $p(g) \in L^2(\Omega)$, $\mathbf{u}(g) \in H^1(\Omega)^N$, and $\theta(g) \in H^2(\Omega) \cap H_0^1(\Omega)$, it would suffice to have $p_1(g) \in L^2(\Omega)$ because from (7) we would then have $\mathbf{u}_1(g) \in H_0^1(\Omega)^N$ and $\theta_1(g) \in H^2(\Omega) \cap H_0^1(\Omega)$.

To prove (11), consider the Green’s formula

$$\int_{\Omega} \mu \nabla^2 q \, d\Omega = \int_{\Omega} q \nabla^2 \mu \, d\Omega - \int_{\Gamma} q \frac{\partial \mu}{\partial n} \, d\Gamma, \tag{12}$$

which holds for $\mu \in H^2(\Omega) \cap H_0^1(\Omega)$, and $q \in L^2(\Omega)$, such that $\nabla^2 q \in L^2(\Omega)$. Specifically, we choose $q = p_1(g)$ and $\nabla^2 q = \nabla^2 p_1(g) = 0$ so that

$$\int_{\Gamma} \frac{\partial \mu}{\partial n} g \, d\Gamma = \int_{\Omega} p_1(g) \nabla^2 \mu \, d\Omega, \quad \forall \mu \in H^2(\Omega) \cap H_0^1(\Omega). \tag{13}$$

Result (11) follows because the map $\mu \rightarrow \nabla^2 \mu$ has a continuous extension from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$, and the map $\mu \rightarrow \frac{\partial \mu}{\partial n}$ is continuous and surjective taking $H^2(\Omega) \cap H_0^1(\Omega) \rightarrow H^{1/2}(\Omega)$. See [1] for details. ■

2.2. Pressure Boundary Equations

The result in the next theorem is fundamental to the Glowinski-Pironneau method. This result, proven for the case $\eta = 0$ in [10], is, essentially, that for $\mathbf{u}(g)$ and $\theta(g)$ solving (5)–(7),

$$\left. \frac{\partial \theta(g)}{\partial n} \right|_{\Gamma} = 0 \iff \nabla \cdot \mathbf{u}(g) = 0.$$

THEOREM 2.2. Consider $\mathbf{u}(g)$, $\theta_1(g)$, and θ_0 in Lemma 2.1. If $\Gamma \in C^*$, $\mathbf{f} \in L^2(\Omega)^N$, and $g \in H^{-1/2}(\Gamma)$, then in the sense of L^2 derivatives,

$$\left. \frac{\partial \theta_1(g)}{\partial n} \right|_{\Gamma} = - \left. \frac{\partial \theta_0}{\partial n} \right|_{\Gamma} \tag{14}$$

if and only if

$$\nabla \cdot \mathbf{u}(g) = 0.$$

PROOF. If $g \in H^{-1/2}(\Gamma)$, then $\theta(g) \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\nabla \cdot \mathbf{u}(g) = 0 \Rightarrow \theta(g) = 0 \Rightarrow \left. \frac{\partial \theta(g)}{\partial n} \right|_{\Gamma} = 0.$$

Taking distributional derivatives,

$$\begin{aligned} \nabla^2 \nabla^2 \theta(g) &= \nabla^2 (\nabla \cdot \mathbf{u}(g)) = \nabla \cdot (\nabla^2 \mathbf{u}(g)) \\ &= -\frac{1}{\nu} \nabla \cdot (-\eta \mathbf{u}(g) + \mathbf{f} - \nabla p(g)) \\ &= \frac{\eta}{\nu} \nabla \cdot \mathbf{u}(g) = \frac{\eta}{\nu} \nabla^2 \theta(g). \end{aligned}$$

This means that if $\left. \frac{\partial \theta(g)}{\partial n} \right|_{\Gamma} = 0$, then $\theta(g)$ satisfies the biharmonic equation

$$\begin{aligned} \nu \nabla^2 \nabla^2 \theta(g) - \eta \nabla^2 \theta(g) &= 0, \\ \theta(g)|_{\Gamma} &= 0, \\ \left. \frac{\partial \theta(g)}{\partial n} \right|_{\Gamma} &= 0. \end{aligned}$$

The bilinear operator here is continuous and coercive on $H^2(\Omega) \cap H_0^1(\Omega)$, and so $\theta(g) = 0$. Therefore, $\nabla^2 \theta(g) = 0$ and as a result, $\nabla \cdot \mathbf{u}(g) = 0$. ■

Note that (14) provides a means of choosing p_{Γ} in (2) so that $\nabla \mathbf{u}(p_{\Gamma}) = 0$. Defining the bilinear form $a(\cdot, \cdot)$ and linear operator $F(\cdot)$, respectively, as

$$a(g_1, g_2) = \int_{\Gamma} \frac{\partial \theta_1(g_1)}{\partial n} g_2 \, d\Gamma, \quad F(g) = - \int_{\Gamma} \frac{\partial \theta_0}{\partial n} g \, d\Gamma,$$

the function $p_{\Gamma} \in H^{-1/2}(\Gamma)/\mathfrak{R}$ must satisfy

$$a(p_{\Gamma}, g) = F(g), \quad \forall g \in \frac{H^{-1/2}(\Gamma)}{\mathfrak{R}}. \tag{15}$$

The next theorem establishes two equivalent forms of (15), which are useful for developing analytical properties of $a(\cdot, \cdot)$ and also lead to a form more convenient for computing.

THEOREM 2.3. *Let $\Gamma \in C^*$, $\mathbf{f} \in L^2(\Omega)^N$, and $\mathcal{B} = H^{-1/2}(\Gamma)/\mathfrak{R}$. If $\hat{p}(g)$ is any function that satisfies $\hat{p}(g) \in L^2(\Omega)$, $\nabla^2 \hat{p}(g) \in L^2(\Omega)$, and $\hat{p}(g)|_\Gamma = g$, then (15) is equivalent to the following two equations for $p_\Gamma \in \mathcal{B}$:*

$$\int_\Omega \hat{p}(g) \nabla \cdot \mathbf{u}_1(p_\Gamma) - \theta_1(p_\Gamma) \nabla^2 \hat{p}(g) \, d\Omega = - \int_\Omega \hat{p}(g) \nabla \cdot \mathbf{u}_0 - \theta_0 \nabla^2 \hat{p}(g) \, d\Omega, \quad \forall g \in \mathcal{B}, \quad (16)$$

$$\int_\Omega \eta \mathbf{u}_1(p_\Gamma) \cdot \mathbf{u}_1(g) + \nu \nabla \mathbf{u}_1(p_\Gamma) : \nabla \mathbf{u}_1(g) \, d\Omega = - \int_\Omega p_1(g) \nabla \cdot \mathbf{u}_0 \, d\Omega, \quad \forall g \in \mathcal{B}, \quad (17)$$

where $\tau : \sigma = \sum_{i,j} \tau_{ij} \sigma_{ij}$ for second-order tensors τ and σ .

PROOF. Because $\theta_1(g) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\mathbf{u}_1(g) \in H_0^1(\Omega)$, Green's second identity leads to

$$\begin{aligned} \int_\Gamma \frac{\partial \theta_1(p_\Gamma)}{\partial n} g \, d\Gamma &= \int_\Omega \hat{p}(g) \nabla^2 \theta_1(p_\Gamma) \, d\Omega - \int_\Omega \theta_1(p_\Gamma) \nabla^2 \hat{p}(g) \, d\Omega \\ &= \int_\Omega \hat{p}(g) \nabla \cdot \mathbf{u}_1(p_\Gamma) \, d\Omega - \int_\Omega \theta_1(p_\Gamma) \nabla^2 \hat{p}(g) \, d\Omega. \end{aligned}$$

Then choosing $\hat{p}(g) = p_1(g)$, we have $\nabla^2 p_1(g) = 0$ and so

$$\begin{aligned} \int_\Gamma \frac{\partial \theta_1(p_\Gamma)}{\partial n} g \, d\Gamma &= \int_\Omega p_1(g) \nabla \cdot \mathbf{u}_1(p_\Gamma) \, d\Omega \\ &= - \int_\Omega \nabla p_1(g) \cdot \mathbf{u}_1(p_\Gamma) \, d\Omega + \int_\Gamma p_1(g) \mathbf{u}_1(p_\Gamma) \cdot \mathbf{n} \, d\Gamma \\ &= - \int_\Omega \nabla p_1(g) \cdot \mathbf{u}_1(p_\Gamma) \, d\Omega \\ &= \int_\Omega (\eta \mathbf{u}_1(g) - \nu \nabla^2 \mathbf{u}_1(g)) \cdot \mathbf{u}_1(p_\Gamma) \, d\Omega. \end{aligned}$$

Therefore,

$$\int_\Gamma \frac{\partial \theta_1(p_\Gamma)}{\partial n} g \, d\Gamma = \int_\Omega \eta \mathbf{u}_1(g) \cdot \mathbf{u}_1(p_\Gamma) + \nu \nabla \mathbf{u}_1(g) : \nabla \mathbf{u}_1(p_\Gamma) \, d\Omega. \quad (18)$$

The right-hand sides of (16),(17) are established in the same fashion. ■

From (18) it is clear that $a(\cdot, \cdot)$ is symmetric positive semidefinite on $H^{-1/2}(\Gamma)$. (Note that symmetry may be lost if boundary conditions other than Dirichlet-type are imposed.) The operator is positive definite provided $g \rightarrow \mathbf{u}_1(g)$ is one-to-one, which is true on $H^{-1/2}(\Gamma)/\mathfrak{R}$. But while it is clear that $\sqrt{a(g, g)} \simeq \|\mathbf{u}_1(g)\|_{1,\Omega}$, recall that the needed continuity and coercivity are related to the $H^{-1/2}(\Gamma)/\mathfrak{R}$ norm. More specifically, it is required that

$$\sqrt{a(g, g)} \simeq \inf_{c \in \mathfrak{R}} \|g + c\|_{-1/2,\Gamma},$$

which is established in the following theorem.

THEOREM 2.4. *If $\Gamma \in C^*$, $\mathbf{f} \in L^2(\Omega)^N$, and $\mathcal{B} = H^{-1/2}(\Gamma)/\mathfrak{R}$, the equation*

$$\int_\Gamma \frac{\partial \theta_1(p_\Gamma)}{\partial n} g \, d\Gamma = - \int_\Gamma \frac{\partial \theta_0}{\partial n} g \, d\Gamma, \quad \forall g \in \mathcal{B}, \quad (19)$$

has a unique solution $p_\Gamma \in \mathcal{B}$ where $(\mathbf{u}(p_\Gamma), p(p_\Gamma)) \in H^1(\Omega) \times L^2(\Omega)$ is a weak solution to the unsteady Stokes problem (1).

In addition,

$$\|\nabla \cdot \mathbf{u}_1(g)\|_{0,\Omega} \simeq \left\| \frac{\partial \theta_1(g)}{\partial n} \right\|_{1/2,\Gamma} \simeq \inf_{c \in \mathfrak{R}} \|g + c\|_{-1/2,\Gamma}. \quad (20)$$

PROOF. From (18) it follows that

$$\int_{\Gamma} \frac{\partial \theta_1(g)}{\partial n} g \, d\Gamma \geq \eta \|u_1(g)\|_{0,\Omega}^2 + \nu |u_1(g)|_{1,\Omega}^2, \tag{21}$$

$$\int_{\Gamma} \frac{\partial \theta_1(p_{\Gamma})}{\partial n} g \, d\Gamma \leq (\eta + \nu) \|u_1(p_{\Gamma})\|_{1,\Omega} \|u_1(g)\|_{1,\Omega}, \tag{22}$$

and considering the weak form of the equation for u_1 in (7),

$$\int_{\Omega} \eta u_1(g) \cdot v + \nu \nabla u_1(g) : \nabla v \, d\Omega = \int_{\Omega} p_1(g + c) \nabla \cdot v \, d\Omega, \quad \forall v \in H_0^1(\Omega)^N, \tag{23}$$

for any $c \in \mathfrak{R}$.

Since Ω is connected, the map $v \rightarrow \nabla \cdot v$ is a continuous surjection taking $H_0^1(\Omega)^N \rightarrow L_0^2(\Omega)$. So given g choose $v \in H_0^1(\Omega)^N$ so that $\nabla \cdot v = p(g) + c_v = p(g + c_v)$ and $\|v\|_{1,\Omega} \leq C_1 \|p_1(g + c_v)\|_{0,\Omega}$. Using this with (23) and (11), and noting that $u_1(g) = u_1(g + c_v)$ we have

$$\begin{aligned} C_1 \max(\eta, \nu) \|u_1(g)\|_{1,\Omega} &\geq \|p_1(g + c_v)\|_{0,\Omega} \geq C_2 \|g + c_v\|_{-1/2,\Gamma} \\ &\geq C_2 \inf_{c \in \mathfrak{R}} \|g + c\|_{-1/2,\Gamma}. \end{aligned}$$

Now, if $\nabla \cdot u_1(g) = 0$, then choose v so that $\nabla \cdot v = 0$, and from (23)

$$\int_{\Omega} \eta u_1(g) \cdot v + \nu \nabla u_1(g) : \nabla v \, d\Omega = \int_{\Omega} p_1(g + c) \nabla \cdot v \, d\Omega = 0.$$

If $\nabla \cdot u_1(g) \neq 0$, then choose $v = u_1(g)$, so that equation (23) leads to

$$\int_{\Omega} \eta u_1(g) \cdot u_1(g) + \nu \nabla u_1(g) : \nabla u_1(g) \, d\Omega = \int_{\Omega} p_1(g + c) \nabla \cdot u_1(g) \, d\Omega.$$

Using this with (11) results in

$$\nu \|u_1(g)\|_{1,\Omega} \leq \inf_{c \in \mathfrak{R}} C_3 \|p_1(g + c)\|_{0,\Omega} \leq C_3 \inf_{c \in \mathfrak{R}} \|g + c\|_{-1/2,\Gamma}.$$

That is,

$$\|u_1(g)\|_{1,\Omega} \simeq \|g + c\|_{-1/2,\Gamma}. \tag{24}$$

Thus $a(\cdot, \cdot)$ is a symmetric, continuous, and coercive bilinear form on $\mathcal{B} \times \mathcal{B}$. So using Lemma 2.1 and the Lax-Milgram theorem, the existence and uniqueness of a solution follows.

Now to establish (20), using Theorem 2.3 and Lemma 2.1, and noting that $\nabla \cdot u_1(g) \in L^2(\Omega)$, gives

$$\begin{aligned} \left(\|q\|_{1/2,\Gamma}\right)^{-1} \int_{\Gamma} \frac{\partial \theta_1(q)}{\partial n} q \, d\Gamma &= \left(\|q\|_{1/2,\Gamma}\right)^{-1} \int_{\Omega} p_1(q) \nabla \cdot u_1(g) \, d\Omega \\ &\leq C_4 \left(\|p_1(q)\|_{0,\Omega}\right)^{-1} \int_{\Omega} p_1(q) \nabla \cdot u_1(g) \, d\Omega \\ &\leq C_4 \|\nabla \cdot u_1(g)\|_0, \end{aligned}$$

which holds $\forall q \in H^{-1/2}$. Using this and (24) gives

$$\left\| \frac{\partial \theta_1(g)}{\partial n} \right\|_{1/2,\Gamma} \leq C_4 \|\nabla \cdot u_1(g)\|_{0,\Omega} \leq C_5 \|u_1(g)\|_{1,\Omega} \leq C_6 \inf_{c \in \mathfrak{R}} \|g + c\|_{-1/2,\Gamma}^2.$$

Finally, as $a_{\Gamma}(\cdot, \cdot)$ is coercive, we have

$$\begin{aligned} \int_{\Gamma} \frac{\partial \theta_1(g)}{\partial n} g \, d\Gamma &\geq C_7 \inf_{c \in \mathfrak{R}} \|g + c\|_{-1/2,\Gamma}^2 \\ \Rightarrow \left\| \frac{\partial \theta_1(g)}{\partial n} \right\|_{1/2,\Gamma} &\geq C_7 \inf_{c \in \mathfrak{R}} \|g + c\|_{-1/2,\Gamma}. \end{aligned} \quad \blacksquare$$

Note that while Theorem 2.4 establishes that the boundary pressure is uniquely determined, it also demonstrates how perturbations of p_{Γ} affect $\nabla \cdot u(p_{\Gamma}) = \nabla \cdot (u_0 + u_1(p_{\Gamma}))$. Specifically, small changes in the boundary pressure p_{Γ} result in small changes in $\nabla \cdot u(p_{\Gamma})$.

3. VARIATIONAL FORMULATION

Solving either (16) or (17) in their current form would be awkward, at the least, for several reasons. The choice of (17) would require that for each pair of boundary functions (g_i, g_j) , $\mathbf{u}_1(g_i)$ and $\mathbf{u}_1(g_j)$ must be used simultaneously. The use of (16) with $\{\theta_1(g), \theta_0\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ implies an especially large linear system. However, working in the subspace $\tilde{\mathcal{B}} = H^{1/2}(\Gamma)/\mathfrak{R}$ of \mathcal{B} provides the benefit of increasing the regularity of $p_1(g)$ and relaxing the regularity required of $\theta_1(g)$ and θ_0 . Also, like (16), the boundary pressure equation does not explicitly require using $p_1(g)$, as illustrated in the following lemma.

LEMMA 3.1. *If $\Gamma \in C^*$ and $g \in \tilde{\mathcal{B}} = H^{1/2}(\Gamma)/\mathfrak{R}$ then $p_1(g)$ is in $H^1(\Omega)$ and is characterized by*

$$\int_{\Omega} \nabla p_1(g) \cdot \nabla \mu \, d\Omega = 0, \quad \forall \mu \in H_0^1(\Omega), \quad p_1(g)|_{\Gamma} = g, \quad (25)$$

and (16) can be written as

$$\int_{\Omega} \hat{p}(g) \nabla \cdot \mathbf{u}_1(p_{\Gamma}) + \nabla \theta_1(p_{\Gamma}) \cdot \nabla \hat{p}(g) \, d\Omega = - \int_{\Omega} \hat{p}(g) \nabla \cdot \mathbf{u}_0 + \nabla \theta_0 \cdot \nabla \hat{p}(g) \, d\Omega, \quad (26)$$

where $\hat{p}(g)$ is any function in $H^1(\Omega)$ that satisfies $\hat{p}(g)|_{\Gamma} = g$.

PROOF. The equivalence of boundary pressure equation (26) is clear from Green's first identity provided $p_1(g) \in H^1(\Omega)$, and this condition is established by considering the variational form (13) as a map taking $g \rightarrow p_1(g)$. Because $\Gamma \in C^*$, for a given g we know that (25) has a unique solution $p_1(g) \in H^1(\Omega)$. Because (25) holds for all $\mu \in H_0^1(\Omega)$, it also holds for all $\mu \in H^2(\Omega) \cap H_0^1(\Omega)$. Thus the unique solution to (25) corresponds to the unique solution to (13). ■

3.1. Variational Formulation in Infinite-Dimensional Spaces

Lemma 3.1 shows that when seeking a solution p_{Γ} in the closed subspace $\tilde{\mathcal{B}}$ of \mathcal{B} , it is only necessary to require that $\{\theta_1(p_{\Gamma}), \theta_0\} \subset H_0^1(\Omega)$. This allows for the approximation of $\theta_1(g)$, θ_0 , $p_1(g)$, and p_0 using the same finite-element subspaces. Lemma 3.1 also implies that $\hat{p}(g)$ can be chosen to be nonzero only in the vicinity of the boundary.

The variational formulation is as follows.

PROBLEM 3.2. *Given $\Gamma \in C^*$, $\mathcal{B} = H^{-1/2}(\Gamma)$, and $\mathbf{f} \in L^2(\Omega)$, find a weak solution $(\mathbf{u}(p_{\Gamma}), p(p_{\Gamma}))$ for (1) along with auxiliary variables p_{Γ} and $\theta(p_{\Gamma})$, as follows: determine p_0 , \mathbf{u}_0 , and θ_0 so that $p_0|_{\Gamma} = 0$, $\mathbf{u}_0|_{\Gamma} = \mathbf{b}$, $\theta_0|_{\Gamma} = 0$, and*

$$\int_{\Omega} \nabla p_0 \cdot \nabla \mu \, d\Omega = \int_{\Omega} \mathbf{f} \cdot \nabla \mu \, d\Omega, \quad \forall \mu \in H_0^1(\Omega), \quad (27)$$

$$\int_{\Omega} \eta \mathbf{u}_0 \cdot \mathbf{v} + \nu \nabla \mathbf{u}_0 : \nabla \mathbf{v} \, d\Omega = \int_{\Omega} (\mathbf{f} - \nabla p_0) \cdot \mathbf{v} \, d\Omega, \quad \forall \mathbf{v} \in H_0^1(\Omega)^N, \quad (28)$$

$$\int_{\Omega} \nabla \theta_0 \cdot \nabla \mu \, d\Omega = \int_{\Omega} (-\nabla \cdot \mathbf{u}_0) \mu \, d\Omega, \quad \forall \mu \in H_0^1(\Omega). \quad (29)$$

Determine $p_{\Gamma} \in \mathcal{B}$ so that

$$\begin{aligned} & \int_{\Omega} p_1(g) \nabla \cdot \mathbf{u}_1(p_{\Gamma}) + \nabla \theta_1(p_{\Gamma}) \cdot \nabla p_1(g) \, d\Omega \\ &= - \int_{\Omega} p_1(g) \nabla \cdot \mathbf{u}_0 + \nabla \theta_0 \cdot \nabla p_1(g) \, d\Omega, \quad \forall g \in \mathcal{B}, \end{aligned} \quad (30)$$

where, given $g \in \mathcal{B}$, the functions $p_1(g)$, $\mathbf{u}_1(g)$, and $\theta_1(g)$ are determined so that $p_1(g)|_\Gamma = g$, $\mathbf{u}_1(g)|_\Gamma = 0$, $\theta_1(g)|_\Gamma = 0$, and

$$\int_\Omega \nabla p_1(g) \cdot \nabla \mu \, d\Omega = 0, \quad \forall \mu \in H^2(\Omega) \cap H_0^1(\Omega), \quad (31)$$

$$\int_\Omega \eta \mathbf{u}_1(g) \cdot \mathbf{v} + \nu \nabla \mathbf{u}_1(g) : \nabla \mathbf{v} \, d\Omega = \int_\Omega -\nabla p_1(g) \cdot \mathbf{v} \, d\Omega, \quad \forall \mathbf{v} \in H_0^1(\Omega)^N, \quad (32)$$

$$\int_\Omega \nabla \theta_1(g) \cdot \nabla \mu \, d\Omega = \int_\Omega (-\nabla \cdot \mathbf{u}_1(g)) \mu \, d\Omega, \quad \forall \mu \in H_0^1(\Omega). \quad (33)$$

The solution is given by

$$\mathbf{u}(p_\Gamma) = \mathbf{u}_0 + \mathbf{u}_1(p_\Gamma), \quad (34)$$

$$p(p_\Gamma) = p_0 + p_1(p_\Gamma). \quad (35)$$

THEOREM 3.3. *If $\Gamma \in C^*$, $\mathbf{f} \in L^2(\Omega)^N$, and $\mathcal{B} = H^{-1/2}(\Gamma)/\mathfrak{R}$, then the variational formulation given in (27)–(35) has a unique solution $(p_\Gamma, p(p_\Gamma), \mathbf{u}(p_\Gamma)) \in \mathcal{B} \times H^1(\Omega) \times H^1(\Omega)^N$.*

PROOF. By Lemma 2.1, each of the Poisson problems has a unique solution. From Theorem 2.4 we know that (30) has a unique solution. So Theorem 3.3 follows immediately. ■

REMARK 3.4. When working within a closed subspace of $H^{-1/2}(\Gamma)$, note that $\mathbf{u}(p_\Gamma)$ may not be the same as \mathbf{u} , the standard weak solution of (1) because the divergence free condition, $\nabla \cdot \mathbf{u}(p_\Gamma) = 0$, is imposed differently.

3.2. Variational Formulation in Finite-Dimensional Spaces

Assume that $\bar{\Omega}$ is a convex planar region and that \mathcal{T} is a triangulation of Ω with interior nodes $a_{n,\Omega}$ and boundary nodes $a_{n,\Gamma}$. Set the pressure at node $a_{0,\Gamma}$ so that the pressure solution is uniquely determined. This will not interfere with using the same space to estimate θ_1 and θ_0 because they are defined as zero on Γ . Finally, since it is necessary to solve for the boundary pressure, the pressure space is decomposed by separating the basis functions along the boundary from those that are strictly interior. The Taylor-Hood finite-element spaces—continuous piecewise quadratic functions for velocity and continuous piecewise linear functions for pressure—will be used. In light of the above conditions, define the following finite-element spaces:

$$\begin{aligned} X_h &= \left\{ \mathbf{v} \in C^0(\bar{\Omega})^2 : \mathbf{v}|_e \in P_2^2, \forall e \in \mathcal{T}_h \right\}, \\ V_h &= X_h \cap H_0^1(\Omega), \\ W_h &= (X_h - V_h) \cup \{0\}, \\ Q_h &= \left\{ q_h \in C^0(\bar{\Omega})^2 : q|_e \in P_1, \forall e \in \mathcal{T}_h \text{ and } q(a_{0,\Gamma}) = 0 \right\}, \\ G_h &= \{q \in Q_h : q(a_{n,\Omega}) = 0, \forall n\}, \\ \Phi_h &= Q_h \cap H_0^1(\Omega). \end{aligned}$$

For Q_h in particular, and for G_h or Φ_h as applicable, the norm $\|q\|_{0/\mathfrak{R}} = \inf_{c \in \mathfrak{R}} \|q + c\|_0$ will be used. For the sake of notation, the basis functions associated with these spaces are as follows:

$$\begin{aligned} V_h &= \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots \}, \\ W_h &= \text{span} \{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \dots \}, \\ Q_h &= \text{span} \{ q_1, q_2, q_3, \dots \}, \\ G_h &= \text{span} \{ g_1, g_2, g_3, \dots \}, \\ \Phi_h &= \text{span} \{ \phi_1, \phi_2, \phi_3, \dots \}. \end{aligned}$$

Note that the functions in G_h are defined on $\bar{\Omega}$, and they have support limited closely along the boundary. Also note that $g_i \in G_h \subset H^1(\Omega) \Rightarrow g_i|_\Gamma \in H^{1/2}(\Gamma)$. So to represent the pressure on the boundary, let $\mathcal{B}_h = \text{span}\{g_i|_\Gamma : g_i \in G_h\}$. This gives $\mathcal{B}_h \subset \mathcal{B}$ as required by Lemma 3.1, so (26) may be used with $\hat{p}(g_i|_\Gamma) = g_i$ along with appropriate choices for $\theta_1(g_i|_\Gamma)$ and θ_0 .

Given these spaces, the discrete variational formulation is as follows.

PROBLEM 3.5. Given $\mathbf{f} \in L^2(\Omega)$, find an approximate solution $(\mathbf{u}_h(p_\Gamma), p_h(p_\Gamma))$ for (1) along with auxiliary variables p_Γ and $\theta_h(p_\Gamma)$, as follows.

STEP 1. Determine $p_{h0} \in \Phi_h$, $\mathbf{u}_{h0} \in X_h$, and $\theta_{h0} \in \Phi_h$ so that $p_{h0}|_\Gamma = 0$, $\mathbf{u}_{h0}|_\Gamma = \mathbf{b}$, $\theta_{h0}|_\Gamma = 0$, and

$$\int_\Omega \nabla p_{h0} \cdot \nabla \phi \, d\Omega = \int_\Omega \mathbf{f} \cdot \nabla \phi \, d\Omega, \quad \forall \phi \in \Phi_h, \tag{36}$$

$$\int_\Omega \eta \mathbf{u}_{h0} \cdot \mathbf{v} + \nu \nabla \mathbf{u}_{h0} : \nabla \mathbf{v} \, d\Omega = \int_\Omega (\mathbf{f} - \nabla p_{h0}) \cdot \mathbf{v} \, d\Omega, \quad \forall \mathbf{v} \in V_h, \tag{37}$$

$$\int_\Omega \nabla \theta_{h0} \cdot \nabla \phi \, d\Omega = \int_\Omega (-\nabla \cdot \mathbf{u}_{h0}) \phi \, d\Omega, \quad \forall \phi \in \Phi_h. \tag{38}$$

STEP 2. Determine $p_\Gamma = \sum \alpha_j g_j \in G_h$ so that

$$\begin{aligned} & \int_\Omega g \nabla \cdot \mathbf{u}_{h1}(p_\Gamma) + \nabla \theta_{h1}(p_\Gamma) \cdot \nabla g \, d\Omega \\ &= - \int_\Omega g \nabla \cdot \mathbf{u}_{h0} + \nabla \theta_{h0} \cdot \nabla g \, d\Omega, \quad \text{for } g = g_i \in G_h, \quad i = 1, \dots \end{aligned} \tag{39}$$

STEP 2.1. As needed, for $g \in G_h$, determine $p_{h1}(g) \in Q_h$, $\mathbf{u}_{h1}(g) \in V_h$, and $\theta_{h1}(g) \in \Phi_h$ so that $p_{h1}(g)|_\Gamma = g|_\Gamma$, $\mathbf{u}_{h1}(g)|_\Gamma = 0$, $\theta_{h1}(g)|_\Gamma = 0$, and

$$\int_\Omega \nabla p_{h1}(g) \cdot \nabla \phi \, d\Omega = 0, \quad \forall \phi \in \Phi_h, \tag{40}$$

$$\int_\Omega \eta \mathbf{u}_{h1}(g) \cdot \mathbf{v} + \nu \nabla \mathbf{u}_{h1}(g) : \nabla \mathbf{v} \, d\Omega = \int_\Omega -\nabla p_{h1}(g) \cdot \mathbf{v} \, d\Omega, \quad \forall \mathbf{v} \in V_h, \tag{41}$$

$$\int_\Omega \nabla \theta_{h1}(g) \cdot \nabla \phi \, d\Omega = \int_\Omega (-\nabla \cdot \mathbf{u}_{h1}(g)) \phi \, d\Omega, \quad \forall \phi \in \Phi_h. \tag{42}$$

STEP 3. The solution is given by

$$p_h(p_\Gamma) = p_{h0} + p_{h1}(p_\Gamma), \tag{43}$$

$$\mathbf{u}_h(p_\Gamma) = \mathbf{u}_{h0} + \mathbf{u}_{h1}(p_\Gamma), \tag{44}$$

$$\theta_h(p_\Gamma) = \theta_{h0} + \theta_{h1}(p_\Gamma). \tag{45}$$

REMARK 3.6. In implementing the algorithm, Step 2 solves for the coefficients α_j , $j = 1, \dots$, using the system of equations

$$\sum_j \alpha_j \int_\Omega g_i \nabla \cdot \mathbf{u}_{h1}(g_j) + \nabla \theta_{h1}(g_j) \cdot \nabla g_i \, d\Omega = - \int_\Omega g_i \nabla \cdot \mathbf{u}_{h0} + \nabla \theta_{h0} \cdot \nabla g_i \, d\Omega, \quad i = 1, \dots$$

Depending upon storage capabilities, $p_{h1}(p_\Gamma)$, $\mathbf{u}_{h1}(p_\Gamma)$, and $\theta_{h1}(p_\Gamma)$ might be formed as linear combinations of stored vectors $p_{h1}(g_j)$, $\mathbf{u}_{h1}(g_j)$, and $\theta_{h1}(g_j)$, $j = 1, \dots$. As storage becomes an issue, these vectors can be discarded after use, and $p_{h1}(p_\Gamma)$, $\mathbf{u}_{h1}(p_\Gamma)$, and $\theta_{h1}(p_\Gamma)$ can be calculated as in Step 2.1 using $g = p_\Gamma$.

Furthermore, the stiffness matrices associated with (36) and (40) are identical, as is the case with (37) and (41), and (38) and (42).

Note that in the discrete variational formulation the boundary equation as given in Lemma 3.1 has been imposed, and substituting $g \in G_h$ for $\hat{p}(g)$ does satisfy the constraints of that lemma. However, p_0 , $p_1(g)$, θ_0 , and $\theta_1(g)$ are variational solutions in closed subspaces of $H^1(\Omega)$, so it is necessary to verify that the pressure boundary equation is equivalent to (17).

LEMMA 3.7. *The boundary equation (39) is equivalent to*

$$\int_{\Omega} \eta \mathbf{u}_{h1}(g) \cdot \mathbf{u}_{h1}(p_{\Gamma}) + \nu \nabla \mathbf{u}_{h1}(g) : \nabla \mathbf{u}_{h1}(p_{\Gamma}) \, d\Omega = \int_{\Omega} p_{h1}(g) \nabla \cdot \mathbf{u}_{h0} \, d\Omega. \quad (46)$$

PROOF. Using (40) with the fact that $\theta_{h1}(p_{\Gamma}) \in \Phi_h$, (42) with $g - p_{h1}(g) \in \Phi_h$, and (41) with $\mathbf{u}_{h1}(p_{\Gamma}) \in V_h$, it follows that

$$\begin{aligned} & \int_{\Omega} g \nabla \cdot \mathbf{u}_{h1}(p_{\Gamma}) + \nabla \theta_{h1}(p_{\Gamma}) \cdot \nabla g \, d\Omega \\ &= \int_{\Omega} g \nabla \cdot \mathbf{u}_{h1}(p_{\Gamma}) + \nabla \theta_{h1}(p_{\Gamma}) \cdot (\nabla g - \nabla p_{h1}(g)) \, d\Omega \\ &= \int_{\Omega} g \nabla \cdot \mathbf{u}_{h1}(p_{\Gamma}) + \mathbf{u}_{h1}(p_{\Gamma}) \cdot (\nabla g - \nabla p_{h1}(g)) \, d\Omega \\ &= - \int_{\Omega} \mathbf{u}_{h1}(p_{\Gamma}) \cdot \nabla p_{h1}(g) \, d\Omega \\ &= \int_{\Omega} \eta \mathbf{u}_{h1}(g) \cdot \mathbf{u}_{h1}(p_{\Gamma}) + \nu \nabla \mathbf{u}_{h1}(g) : \nabla \mathbf{u}_{h1}(p_{\Gamma}) \, d\Omega. \end{aligned}$$

Similarly,

$$\begin{aligned} & - \int_{\Omega} g \nabla \cdot \mathbf{u}_{h0} + \nabla \theta_{h0} \cdot \nabla g \, d\Omega \\ &= - \int_{\Omega} g \nabla \cdot \mathbf{u}_{h0} + \nabla \theta_{h0} \cdot (\nabla g - \nabla p_{h1}(g)) \, d\Omega \\ &= - \int_{\Omega} g \nabla \cdot \mathbf{u}_{h0} + \mathbf{u}_{h0} \cdot (\nabla g - \nabla p_{h1}(g)) \, d\Omega \\ &= \int_{\Omega} \mathbf{u}_{h0} \cdot \nabla p_{h1}(g) \, d\Omega - \int_{\Gamma} g \mathbf{u}_{h0} \cdot \mathbf{n} \, d\Gamma \\ &= \int_{\Omega} p_{h1}(g) \nabla \cdot \mathbf{u}_{h0} \, d\Omega, \end{aligned}$$

and the result follows. ■

Now it is shown that the discretization has a unique solution.

THEOREM 3.8. *The bilinear form in the boundary equation (39) is symmetric positive definite on $G_h \times G_h$, and Problem 3.5 has a unique solution*

$$(p_{\Gamma}, p_h(p_{\Gamma}), \mathbf{u}_h(p_{\Gamma}), \theta_h(p_{\Gamma})) \in G_h \times Q_h \times X_h \times \Phi_h. \quad (47)$$

PROOF. It is clear that the Poisson problems have a unique solution and that the pressure and velocity discrete operators are symmetric positive definite. The discrete boundary pressure equation operator is at least symmetric positive indefinite by Lemma 3.7. Now it is shown that this operator is positive definite. Note that

$$\begin{aligned} & \int_{\Omega} g \nabla \cdot \mathbf{u}_{h1}(p_{\Gamma}) + \nabla \theta_{h1}(p_{\Gamma}) \cdot \nabla g \, d\Omega \\ &= \int_{\Omega} \eta \mathbf{u}_{h1}(g) \cdot \mathbf{u}_{h1}(g) + \nu \nabla \mathbf{u}_{h1}(g) : \nabla \mathbf{u}_{h1}(g) \, d\Omega \\ &\geq \nu |\mathbf{u}_{h1}(g)|^2. \end{aligned}$$

If $\mathbf{u}_{h1}(g) = \mathbf{0}$, then (40) implies

$$\int_{\Omega} \nabla p_{h1}(g) \cdot \mathbf{v} \, d\Omega = 0, \quad \forall \mathbf{v} \in V_h. \quad (48)$$

This implies that $p_{h1}(g) = 0$ and so (as will be shown) $g = 0$. That is, the variational form is positive definite. To establish that $p_{h1}(g) = 0$, note that the Taylor-Hood element satisfies the inf-sup condition [2]

$$\sup_{\mathbf{v} \in V_h} \frac{\int_{\Omega} \nabla \tilde{q} \cdot \mathbf{v} \, d\Omega}{|\mathbf{v}|_{1,\Omega}} \geq \omega \|\tilde{q}\|_{0,\Omega}, \quad \forall \tilde{q} \in \tilde{Q}_h. \tag{49}$$

The pressure space is denoted \tilde{Q}_h instead of Q_h because the uniqueness condition is enforced here by insisting $\int_{\Omega} \tilde{q} \, d\Omega = 0$ as opposed to setting the pressure at one node on the boundary. This condition suffices in this case because for each $q \in Q_h$ there exists some constant c_q , such that $q + c_q \in \tilde{Q}_h$, and

$$\sup_{\mathbf{v} \in V_h} \frac{\int_{\Omega} \nabla q \cdot \mathbf{v} \, d\Omega}{|\mathbf{v}|_{1,\Omega}} = \sup_{\mathbf{v} \in V_h} \frac{\int_{\Omega} \nabla (q + c_q) \cdot \mathbf{v} \, d\Omega}{|\mathbf{v}|_{1,\Omega}} \geq \omega \|q + c_q\|_{0,\Omega} \geq \omega \|q\|_{0/\mathbb{R}}.$$

So choosing $q = p_{h1}(g)$, (48) implies $\|p_{h1}(g)\|_{0/\mathbb{R}} = 0$, and so $g = 0$. That is, the map $g \rightarrow \mathbf{u}_{h1}(g)$ taking $G_h \rightarrow V_h$ is one to one, and the proof is complete. ■

REMARK 3.9. The discussion following equation (48) establishes the equivalence of imposing a zero-mean condition and setting the pressure at a point in order to have a unique solution. The former condition is often used in analysis while the latter is imposed on the computed solution.

Returning to the specifics of the discretization, p_{Γ} is approximated as $p_{h\Gamma} = \sum_{j=0}^{|G_h|} \alpha_j g_j$, where the α_j satisfy

$$\begin{aligned} & \sum_{j=0}^{|G_h|} \alpha_j \int_{\Omega} \nabla \theta_1(g_j|_{\Gamma}) \cdot \nabla g_i + g_i \nabla \cdot \mathbf{u}_1(g_j|_{\Gamma}) \, d\Omega \\ &= - \int_{\Omega} \nabla \theta_0 \cdot \nabla g_i + g_i \nabla \cdot \mathbf{u}_0 \, d\Omega, \quad \forall i \in \{1 \dots |G_h|\}. \end{aligned} \tag{50}$$

For each g_j , let

$$\begin{aligned} p_1(g_j|_{\Gamma}) &\approx p_{h1}^j = g_j + \sum_{m=1}^{|\Phi_h|} \beta_m^j \phi_m, & \mathbf{u}_1(g_j|_{\Gamma}) &\approx \mathbf{u}_{h1}^j = \sum_{m=1}^{|\Phi_h|} \gamma_m^j \mathbf{v}_m, \\ \theta_1(g_j|_{\Gamma}) &\approx \theta_{h1}^j = \sum_{m=1}^{|\Phi_h|} \zeta_m^j \phi_m, \end{aligned}$$

where the $\beta^j, \gamma^j, \zeta^j$ are obtained by considering the definitions of $p_1(g_j|_{\Gamma}), \mathbf{u}_1(g_j|_{\Gamma})$, and $\theta_1(g_j|_{\Gamma})$, and insisting that the following hold:

$$\sum_{m=1}^{|\Phi_h|} \beta_m^j \int_{\Omega} (\nabla \phi_m \cdot \nabla \phi_n) \, d\Omega = \int_{\Omega} -\nabla g_j \cdot \nabla \phi_n \, d\Omega, \quad \forall n \in \{1 \dots |\Phi_h|\}, \tag{51}$$

$$\sum_{m=1}^{|\Phi_h|} \gamma_m^j \int_{\Omega} (\eta \mathbf{v}_m \cdot \mathbf{v}_n + \nu \nabla \mathbf{v}_m : \nabla \mathbf{v}_n) \, d\Omega = \int_{\Omega} -\nabla p_1^j \cdot \mathbf{v}_n \, d\Omega, \quad \forall n \in \{1 \dots |\Phi_h|\}, \tag{52}$$

$$\sum_{m=1}^{|\Phi_h|} \zeta_m^j \int_{\Omega} (\nabla \phi_m \cdot \nabla \phi_n) \, d\Omega = - \int_{\Omega} \phi_n \nabla \cdot \mathbf{u}_{1h}^j \, d\Omega, \quad \forall n \in \{1 \dots |\Phi_h|\}. \tag{53}$$

The values of p_0, \mathbf{u}_0 , and θ_0 are determined similarly. Let

$$p_0 \approx p_{h0} = \sum_{m=1}^{|\Phi_h|} \beta_m^0 \phi_m, \quad \mathbf{u}_0 \approx \mathbf{u}_{h0} = \sum_{m=1}^{|\Phi_h|} \gamma_m^0 \mathbf{v}_m + \mathbf{b}_h, \quad \theta_0 \approx \theta_{h0} = \sum_{m=1}^{|\Phi_h|} \zeta_m^0 \phi_m,$$

where $\mathbf{b}_h = \sum_{k=1}^{|W_h|} \delta_k^0 \mathbf{w}_k \in W_h$ and the δ_k^0 are selected so that $\mathbf{b}_h|_\Gamma$ interpolates \mathbf{b} . Then the following equations must hold:

$$\sum_{m=1}^{|\Phi_h|} \beta_m^0 \int_{\Omega} (\nabla \phi_m \cdot \nabla \phi_n) \, d\Omega = \int_{\Omega} \mathbf{f} \cdot \nabla \phi_n \, d\Omega, \quad \forall n \in \{1 \cdots |\Phi_h|\}, \quad (54)$$

$$\begin{aligned} \sum_{m=1}^{|V_h|} \gamma_m^0 \int_{\Omega} (\eta \mathbf{v}_m \cdot \mathbf{v}_n + \nu \nabla \mathbf{v}_m : \nabla \mathbf{v}_n) \, d\Omega \\ = \int_{\Omega} (\mathbf{f} - \nabla p_{h0} - \eta \mathbf{b}_h) \\ \cdot \mathbf{v}_n - \nu \nabla \mathbf{b}_h : \nabla \mathbf{v}_n \, d\Omega, \quad \forall n \in \{1 \cdots |V_h|\}, \end{aligned} \quad (55)$$

$$\sum_{m=1}^{|\Phi_h|} \zeta_m^0 \int_{\Omega} (\nabla \phi_m \cdot \nabla \phi_n) \, d\Omega = - \int_{\Omega} \phi_n \nabla \cdot \mathbf{u}_{h0}^j \, d\Omega, \quad \forall n \in \{1 \cdots |\Phi_h|\}. \quad (56)$$

Note that the process of determining p_Γ involves three discrete linear operators. For notation, call these A_{bp} , A_{ip} , and A_v for boundary pressure (50), interior pressure (51), (53), (54), (56), and velocity (52),(55). It is important to note that none of these operators depend upon \mathbf{f} or \mathbf{b} .

Note also that A_v can be permuted to the form $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ through judicious selection/ordering of basis functions. So the effective size of A_v is $|V_h|/2$.

The size of A_{bp} depends upon the number of nodes along the boundary and so the system is relatively small. However, the calculation of each entry of A_{bp} requires solving two systems involving A_{ip} , and one system involving A_v . So if one should use this approach in an iterative scheme such as the θ -method, it is fortunate that A_{bp} need only be recalculated after changes in the mesh. That is, \mathbf{f} and \mathbf{b} may be changed without effecting A_{bp} .

Finally, note that all entries of A_{bp} are independent of each other, and so they can be calculated in a parallel fashion.

4. ERROR ANALYSIS

THEOREM 4.1. *Suppose that $p(\text{mod } \mathfrak{R})$ and \mathbf{u} comprise the true solution to the unsteady Stokes problem (1) with $\mathbf{b} = \mathbf{0}$, and $\bar{\Omega}$ a convex polygon with a regular triangulation. If $(p, \mathbf{u}) \in H^{k+1}(\Omega)^2 \times H^k(\Omega)(\text{mod } \mathfrak{R})$ for $k \in \{1, 2\}$ then the unique solution $(p_h, \mathbf{u}_h, \theta_h) = (p_h(p_\Gamma), \mathbf{u}_h(p_\Gamma), \theta_h(p_\Gamma)) \in Q_h \times X_h \times \Phi_h$ of Problem 3.5 satisfies the following error bounds:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_1 &\leq Ch^k (|\mathbf{u}|_{k+1} + |p|_k), \\ |\theta_h|_1 &\leq \|\mathbf{u} - \mathbf{u}_h\|_0, \\ \|p - p_h\|_{0/\mathfrak{R}} &\leq Ch^k (|\mathbf{u}|_{k+1} + |p|_k), \\ \|\mathbf{u} - \mathbf{u}_h + \nabla \theta_h\|_0 &\leq Ch^{k+1} (|\mathbf{u}|_{k+1} + |p|_k). \end{aligned}$$

PROOF. The proof of the error bounds will make use of the two equalities

$$\begin{aligned} \eta (\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \nu (\nabla (\mathbf{u} - \mathbf{u}_h), \nabla \mathbf{v}_h) \\ + (\nabla (p - p_h), \mathbf{v}_h - \nabla \phi_h) = 0, \quad \forall (\mathbf{v}_h, \phi_h) \in V_h \times \Phi_h, \end{aligned} \quad (57)$$

$$(\nabla \theta_h, \nabla q_h) = (\mathbf{u}_h, \nabla q_h), \quad \forall q_h \in Q_h. \quad (58)$$

To establish (57) and (58), note that the following equalities hold by construction:

$$(\nabla p_h, \nabla \phi_h) = (\mathbf{f}, \nabla \phi_h), \quad \forall \phi_h \in \Phi_h, \quad (59)$$

$$\eta (\mathbf{u}_h, \mathbf{v}_h) + \nu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) = (\mathbf{f} - \nabla p_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h, \quad (60)$$

$$(\nabla \theta_h, \nabla \phi_h) = (\mathbf{u}_h, \nabla \phi_h), \quad \forall \phi_h \in \Phi_h. \quad (61)$$

Equation (57) follows from (59)–(61) by observing that p , \mathbf{u} , and θ also satisfy (59)–(61). Also, (58) holds by extending (61) to hold on Q_h . With this in mind, given $q_h \in Q_h$, note that there is a unique $g_h \in G_h$, such that $q_h|_\Gamma = g_h|_\Gamma$, so q_h has a unique representation $q_h = p_{h1}(g_h) + \phi_h$ with $\phi_h \in \Phi_h$ and $p_{h1}(g_h)$ satisfying (40). Uniqueness of this representation follows from the uniqueness of p_{h1} as a function of g_h . This gives

$$\begin{aligned} (\mathbf{u}_h, \nabla q_h) &= (\mathbf{u}_h, \nabla p_{h1}(g_h) + \nabla \phi_h) = (\mathbf{u}_h, \nabla p_{h1}(g_h)) + (\mathbf{u}_h, \nabla \phi_h), \\ (\nabla \theta_h, \nabla q_h) &= (\nabla \theta_h, \nabla p_{h1}(g_h) + \nabla \phi_h) = (\nabla \theta_h, \nabla p_{h1}(g_h)) + (\nabla \theta_h, \nabla \phi_h). \end{aligned}$$

Noting (38) and (42) gives $(\nabla \theta_h, \nabla \phi_h) = (\mathbf{u}_h, \nabla \phi_h)$. Also, $\theta_h \in \Phi_h$, which means (40) gives $(\nabla \theta_h, \nabla p_{h1}(g_h)) = 0$. So to show $(\mathbf{u}_h, \nabla q_h) = (\nabla \theta_h, \nabla q_h)$, it will suffice to show $(\mathbf{u}_h, \nabla p_{h1}(g_h)) = 0$. To do this, recall that \mathbf{u}_h is the approximation, so $\mathbf{u}_h = \mathbf{u}_h(p_\Gamma) \in V_h$. This means using $\mathbf{u}_h(p_\Gamma)$ as \mathbf{v} in (41) gives

$$\begin{aligned} (\mathbf{u}_h(g), \nabla p_{h1}(g_h)) &= -\eta(\mathbf{u}_{h1}(g), \mathbf{u}_{h0} + \mathbf{u}_{h1}(p_\Gamma)) - \nu(\nabla \mathbf{u}_{h1}(g), \nabla \mathbf{u}_{h0} + \nabla \mathbf{u}_{h1}(p_\Gamma)) \\ &= -\eta(\mathbf{u}_{h1}(g), \mathbf{u}_{h0}) - \nu(\nabla \mathbf{u}_{h1}(g), \nabla \mathbf{u}_{h0}) \\ &\quad - \eta(\mathbf{u}_{h1}(g), \mathbf{u}_{h1}(p_\Gamma)) - \nu(\nabla \mathbf{u}_{h1}(g), \nabla \mathbf{u}_{h1}(p_\Gamma)) \\ &= -\eta(\mathbf{u}_{h1}(g), \mathbf{u}_{h0}) - \nu(\nabla \mathbf{u}_{h1}(g), \nabla \mathbf{u}_{h0}) + (\nabla \cdot \mathbf{u}_{h0}, p_{h1}(g)) \tag{62} \\ &= -\eta(\mathbf{u}_{h1}(g), \mathbf{u}_{h0}) - \nu(\nabla \mathbf{u}_{h1}(g), \nabla \mathbf{u}_{h0}) - (\mathbf{u}_{h0}, \nabla p_{h1}(g)) \\ &= (\mathbf{u}_{h0}, \nabla p_{h1}(g)) - (\mathbf{u}_{h0}, \nabla p_{h1}(g)) \tag{63} \\ &= 0. \end{aligned}$$

Equality at (62) follows from the equivalence of the boundary equations established in Lemma 3.7, and using (41) once more shows equality at (63). Therefore, (57) and (58) hold, and may be used to produce the error bounds.

BOUNDS FOR $\mathbf{u} - \mathbf{u}_h$. To produce error estimates for velocity, it is convenient to introduce the space

$$D_h = \{(\mathbf{v}, \phi_h) \in V_h \times \Phi_h : (\mathbf{v}_h - \nabla \phi_h, \nabla q_h) = 0, \forall q_h \in Q_h\}.$$

Note that $D_h \neq \emptyset$, because $(\mathbf{u}_h, \theta_h) \in D_h$. Now restricting the pair (\mathbf{v}_h, ϕ_h) in (57) to D_h , it follows that

$$\begin{aligned} \eta(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla \mathbf{v}_h) + (\nabla(p - q_h), \mathbf{v}_h - \nabla \phi_h) &= 0, \\ \forall ((\mathbf{v}_h, \phi_h), q_h) \in D_h \times Q_h. \end{aligned} \tag{64}$$

For q_h substitute $P_h p$, the projection of p onto Q_h which satisfies

$$(\nabla(p - P_h p), \nabla q_h) = 0, \quad \forall q_h \in Q_h, \quad \text{and} \quad \int_\Omega p \, d\Omega = \int_\Omega P_h p \, d\Omega. \tag{65}$$

That $P_h p$ exists follows from $k \in \{1, 2\}$ and $p \in H^k(\Omega)$. Also note that

$$\|p - P_h p\|_0 = \|p - P_h p\|_{0/\mathcal{R}}.$$

Substituting $\mathbf{u} - \mathbf{u}_h = \mathbf{u} - \mathbf{w}_h + \mathbf{w}_h - \mathbf{u}_h$ in (64) results in

$$\begin{aligned} &\eta(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) + \eta(\mathbf{w}_h - \mathbf{u}_h, \mathbf{v}_h) \\ &+ \nu(\nabla(\mathbf{u} - \mathbf{w}_h), \nabla \mathbf{v}_h) + \nu(\nabla(\mathbf{w}_h - \mathbf{u}_h), \nabla \mathbf{v}_h) \\ &= (p - P_h p, \nabla \cdot \mathbf{v}_h), \quad \forall (\mathbf{v}_h, \phi_h) \in D_h. \end{aligned}$$

Now choose $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h$, and note that $(\mathbf{u}_h, \theta_h) \in D_h$. As a result,

$$\begin{aligned} \eta \|\mathbf{u}_h - \mathbf{w}_h\|_0^2 + \nu |\mathbf{u}_h - \mathbf{w}_h|_1^2 &= \eta (\mathbf{u} - \mathbf{w}_h, \mathbf{u}_h - \mathbf{w}_h) \\ &\quad + \nu (\nabla(\mathbf{u} - \mathbf{w}_h), \nabla(\mathbf{u}_h - \mathbf{w}_h)) \\ &\quad - (\nabla \cdot (\mathbf{u}_h - \mathbf{w}_h), p - P_h p), \quad \forall (\mathbf{w}_h, \varphi_h) \in D_h, \end{aligned}$$

with φ_h dependent upon \mathbf{w}_h . From the above,

$$\begin{aligned} |\mathbf{u}_h - \mathbf{w}_h|_1^2 &\leq \frac{\eta}{\nu} (\mathbf{u} - \mathbf{w}_h, \mathbf{u}_h - \mathbf{w}_h) + (\nabla(\mathbf{u} - \mathbf{w}_h), \nabla(\mathbf{u}_h - \mathbf{w}_h)) \\ &\quad - \frac{1}{\nu} (\nabla \cdot (\mathbf{u}_h - \mathbf{w}_h), p - P_h p), \quad \forall (\mathbf{w}_h, \varphi_h) \in D_h, \\ |\mathbf{u}_h - \mathbf{w}_h|_1 &\leq C_1 \frac{\eta}{\nu} \|\mathbf{u} - \mathbf{w}_h\|_0 + |\mathbf{u} - \mathbf{w}_h|_1 + \frac{\sqrt{2}}{\nu} \|p - P_h p\|_{0/\mathfrak{R}} \\ &\leq \left(1 + C_1 \frac{\eta}{\nu}\right) \|\mathbf{u} - \mathbf{w}_h\|_1 + \frac{\sqrt{2}}{\nu} \|p - P_h p\|_{0/\mathfrak{R}}, \quad \forall (\mathbf{w}_h, \varphi_h) \in D_h. \end{aligned} \tag{66}$$

Now choose $\varphi_h = 0$, and note that $\{\mathbf{v} : (\mathbf{v}, 0) \in D_h\} = \{\mathbf{v} \in V_h : (\nabla \cdot \mathbf{v}, q_h) = 0, \forall q \in Q_h\}$. That is, $\varphi_h = 0$ implies \mathbf{w}_h may be arbitrarily selected from the subspace of V_h whose divergence is orthogonal to Q_h . Therefore, because V_h and Q_h satisfy the inf-sup condition,

$$\sup_{\mathbf{v} \in V_h} \frac{\int_{\Omega} \nabla q \cdot \mathbf{v} \, d\Omega}{|\mathbf{v}|_{1,\Omega}} \geq \omega \|q\|_{0,\mathfrak{R}}, \quad \forall q \in Q_h. \tag{67}$$

The inequality [1]

$$\inf_{(\mathbf{w}, 0) \in D_h} \|\mathbf{u} - \mathbf{w}_h\|_1 \leq \left(1 + \frac{\sqrt{2}}{\omega}\right) \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1$$

may be applied to (66) to obtain a fixed \mathbf{w}_h so that

$$\|\mathbf{u} - \mathbf{w}_h\|_1 \leq \left(1 + \frac{\sqrt{2}}{\omega}\right) \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1. \tag{68}$$

So using (66), it follows that

$$|\mathbf{u}_h - \mathbf{w}_h|_1 \leq \left(1 + C_1 \frac{\eta}{\nu}\right) \left(1 + \frac{\sqrt{2}}{\omega}\right) \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1 + \frac{\sqrt{2}}{\nu} \|p - P_h p\|_{0/\mathfrak{R}}. \tag{69}$$

Because $\mathbf{u} - \mathbf{u}_h = \mathbf{u} - \mathbf{w}_h + \mathbf{w}_h - \mathbf{u}_h$, it follows that

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leq \|\mathbf{u} - \mathbf{w}_h\|_1 + C_2 |\mathbf{u}_h - \mathbf{w}_h|_1$$

which with (68) and (69) gives

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leq C_3 \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1 + C_4 \|p - P_h p\|_{0/\mathfrak{R}}. \tag{70}$$

Now for interpolation estimates on V_h and Q_h , with $k \in \{1, 2\}$ and $n \in \{0, 1\}$,

$$\|\mathbf{v} - I_h \mathbf{v}\|_n \leq C_5 h^{k+1-n} |\mathbf{v}|_{k+1}, \quad \forall \mathbf{v} \in H^{k+1}(\Omega)^2, \quad n \leq k, \tag{71}$$

$$\|p - P_h p\|_{0/\mathfrak{R}} + h \|p - P_h p\|_1 \leq C_6 h^k |p|_k. \tag{72}$$

So from (70), the velocity error bound is

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leq Ch^k (|\mathbf{u}|_{k+1} + |p|_k). \tag{73}$$

BOUNDS FOR θ_h . To bound θ_h , recall from (58) that

$$(\mathbf{u}_h - \nabla\theta_h, \nabla q_h) = 0, \quad \forall q_h \in Q_h,$$

which gives

$$\begin{aligned} (\nabla\theta_h, \nabla\theta_h) &= (\mathbf{u}_h, \nabla\theta_h) \\ &= (\mathbf{u}_h, \nabla\theta_h) - (\nabla \cdot \mathbf{u}, \theta_h) \\ &= -(\mathbf{u} - \mathbf{u}_h, \nabla\theta_h). \end{aligned}$$

So the resulting bound, in terms of the velocity error, is

$$|\theta_h|_1 \leq \|\mathbf{u} - \mathbf{u}_h\|_0.$$

BOUNDS FOR $p - p_h$. To produce bounds for the error in pressure, consider (57) with $\phi_h = 0$. This gives

$$-(\nabla(p - p_h), \mathbf{v}_h) = \eta(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla\mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h,$$

and thus

$$\begin{aligned} (p_h - q_h, \nabla \cdot \mathbf{v}_h) &= -\eta(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla\mathbf{v}_h) \\ &+ (p - q_h, \nabla \cdot \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h, \quad \forall q_h \in Q_h. \end{aligned} \tag{74}$$

Noting $|\nabla \cdot \mathbf{v}|_0 \leq \sqrt{2} \|\mathbf{v}\|_1$, assuming $\mathbf{v} \neq \mathbf{0}$ and dividing both sides of (74) by $\|\mathbf{v}_h\|_1$ gives

$$\frac{(p_h - q_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_1} \leq 2 \max\{\eta, \nu\} \|\mathbf{u} - \mathbf{u}_h\|_1 + \sqrt{2} \|p - q_h\|_0, \quad \forall q_h \in Q_h. \tag{75}$$

Now using the inf-sup condition (67), choose $\mathbf{v}_h \neq \mathbf{0}$ so that

$$\omega \|p_h - q_h\|_{0/\mathfrak{R}} \leq \frac{(p_h - q_h, \nabla \cdot \mathbf{v}_h)}{|\mathbf{v}_h|_1}.$$

Noting that $\|\mathbf{v}_h\|_1 \leq C_1^{-1} |\mathbf{v}_h|_1$, and using (75) gives

$$\|p_h - q_h\|_{0/\mathfrak{R}} \leq C_1 \frac{2}{\omega} \max\{\eta, \nu\} \|\mathbf{u} - \mathbf{u}_h\|_1 + C_1 \frac{\sqrt{2}}{\omega} \|p - q_h\|_{0/\mathfrak{R}}, \quad \forall q_h \in Q_h,$$

so using the triangle inequality

$$\|p - p_h\|_{0/\mathfrak{R}} \leq \|p_h - q_h\|_{0/\mathfrak{R}} + \|p - q_h\|_{0/\mathfrak{R}}$$

results in

$$\|p - p_h\|_{0/\mathfrak{R}} \leq C_1 \frac{2}{\omega} \max\{\eta, \nu\} \|\mathbf{u} - \mathbf{u}_h\|_1 + \left(1 + C_1 \frac{2}{\omega} \sqrt{2}\right) \inf_{q_h \in Q_h} \|p - q_h\|_{0/\mathfrak{R}}, \quad \forall q_h \in Q_h.$$

Now considering (70), substituting $q_h = P_h p$ gives

$$\|p - p_h\|_{0/\mathfrak{R}} \leq C_2 \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1 + C_3 \|p - P_h p\|_{0/\mathfrak{R}},$$

so from the velocity error bound (73),

$$\|p - p_h\|_{0/\mathfrak{R}} \leq Ch^k (|\mathbf{u}|_{k+1} + |p|_k). \tag{76}$$

BOUNDS FOR $\mathbf{u} - \mathbf{u}_h + \nabla\theta_h$. To improve the velocity error bound somewhat, consider a variant of the duality argument of Aubin and Nitsche [11,12]. For a convex polygonal domain, and given $\mathbf{F} \in L^2(\Omega)^2$, there exists a unique solution (ψ, μ) [1] to the Stokes (dual) problem

$$\begin{aligned} -\nabla^2\psi - \nabla\mu &= \mathbf{F}, & \nabla \cdot \psi &= 0, & \psi|_\Gamma &= 0, \\ \psi &\in H^2(\Omega)^2 \cap H_0^1(\Omega)^2, & \mu &\in \frac{H^1(\Omega)}{\mathfrak{R}}, & \|\psi\|_2 + |\mu|_1 &\leq C_1 \|\mathbf{F}\|_0. \end{aligned}$$

To see that this is also true for the unsteady Stokes problem, note that because $\Gamma \in C^*$, by the Lax-Milgram theorem there is a unique solution $\vartheta \in H_0^1(\Omega)^2$ to

$$\eta\vartheta - \nu\nabla^2\vartheta = -\nabla^2\psi, \quad \vartheta|_\Gamma = 0.$$

Noting that $\nu\nabla^2\vartheta = -\nabla^2\psi - \eta\vartheta \in L^2(\Omega)$, it follows from $\Gamma \in C^*$ that $\vartheta \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2$. Taking distributional derivatives gives

$$\nabla \cdot (\eta\vartheta - \nu\nabla^2\vartheta) = \eta\nabla \cdot \vartheta - \nu\nabla^2\nabla \cdot \vartheta = -\nabla \cdot \nabla^2\psi = 0.$$

Now let $\xi = \nabla \cdot \vartheta$ and note that there is a unique solution $\xi = 0$ to

$$\eta\xi - \nu\nabla^2\xi = 0, \quad \xi|_\Gamma = 0.$$

So $\nabla \cdot \vartheta = 0$. It follows that for a given $\mathbf{F} \in L^2(\Omega)^2$, there exists a unique solution (ϑ, μ) to the unsteady Stokes (dual) problem

$$\eta\vartheta - \nu\nabla^2\vartheta - \nabla\mu = \mathbf{F}, \quad \nabla \cdot \vartheta = 0, \quad \vartheta|_\Gamma = 0, \tag{77}$$

$$\vartheta \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2, \quad \mu \in \frac{H^1(\Omega)}{\mathfrak{R}}, \quad \|\vartheta\|_2 + |\mu|_1 \leq C_1 \|\mathbf{F}\|_0. \tag{78}$$

Using the true solution to (77) (ϑ, μ) , the true solution (\mathbf{u}, p) to (1), and the approximation (\mathbf{u}_h, p_h) to (1), gives

$$(\mathbf{F}, \mathbf{u} - \mathbf{u}_h) = \eta(\vartheta, \mathbf{u} - \mathbf{u}_h) + \nu(\nabla\vartheta, \nabla(\mathbf{u} - \mathbf{u}_h)) - (\nabla\mu, \mathbf{u} - \mathbf{u}_h). \tag{79}$$

Also, because (\mathbf{u}_h, p_h) is the approximation for (\mathbf{u}, p) , using (37) and (41) results in

$$0 = \eta(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla\mathbf{v}_h) + (\nabla(p - p_h), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h.$$

Combining this with (58) and the fact that $\nabla \cdot (\eta\vartheta - \nu\nabla^2\vartheta - \nabla\mu) = -\nabla^2\mu = \nabla \cdot \mathbf{F}$,

$$\begin{aligned} 0 &= \eta(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla\mathbf{v}_h) + (\nabla(p - p_h), \mathbf{v}_h) \\ &\quad + (\nabla q_h, \mathbf{u}_h) - (\nabla q_h, \nabla\theta_h) \\ &\quad - (\theta_h, \nabla^2\mu) - (\nabla \cdot \mathbf{F}, \theta_h), \quad \forall \mathbf{v}_h \in V_h, \quad \forall q_h \in Q_h, \\ &= \eta(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla(\mathbf{u} - \mathbf{u}_h), \nabla\mathbf{v}_h) + (\nabla(p - p_h), \mathbf{v}_h) \\ &\quad - (q_h, \nabla \cdot \mathbf{u}_h) + (\nabla\theta_h, \nabla(\mu - q_h)) + (\mathbf{F}, \nabla\theta_h), \quad \forall \mathbf{v}_h \in V_h, \quad \forall q_h \in Q_h. \end{aligned} \tag{80}$$

The equality at (80) follows from the divergence theorem and the fact that $\mathbf{u}_h|_\Gamma = 0$ and $\theta_h|_\Gamma = 0$. Now for q_h choose $P_h\mu$, the projection of μ onto Q_h as in (65). Then subtracting (80) from the right-hand side of (79) gives

$$\begin{aligned}
 (\mathbf{F}, \mathbf{u} - \mathbf{u}_h) &= \eta(\vartheta - \mathbf{v}_h, \mathbf{u} - \mathbf{u}_h) + \nu(\nabla(\vartheta - \mathbf{v}_h), \nabla(\mathbf{u} - \mathbf{u}_h)) \\
 &\quad - (\nabla\mu, \mathbf{u} - \mathbf{u}_h) - (\nabla(p - p_h), \mathbf{v}_h) \\
 &+ (P_h\mu, \nabla \cdot \mathbf{u}_h) - (\nabla\theta_h, \nabla(\mu - P_h\mu)) - (\mathbf{F}, \nabla\theta_h), \quad \forall \mathbf{v}_h \in V_h.
 \end{aligned}
 \tag{81}$$

As in (65), $(\nabla\theta_h, \nabla(\mu - P_h\mu)) = 0$, so using $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \vartheta = 0$ gives

$$\begin{aligned}
 (\mathbf{F}, \mathbf{u} - \mathbf{u}_h + \nabla\theta_h) &= \eta(\vartheta - \mathbf{v}_h, \mathbf{u} - \mathbf{u}_h) + \nu(\nabla(\vartheta - \mathbf{v}_h), \nabla(\mathbf{u} - \mathbf{u}_h)) \\
 &\quad + (\mu, \nabla \cdot (\mathbf{u} - \mathbf{u}_h)) - (p - p_h, \nabla \cdot (\vartheta - \mathbf{v}_h)) \\
 &\quad - (P_h\mu, \nabla \cdot (\mathbf{u} - \mathbf{u}_h))
 \end{aligned}
 \tag{82}$$

$$\begin{aligned}
 &= \eta(\vartheta - \mathbf{v}_h, \mathbf{u} - \mathbf{u}_h) + \nu(\nabla(\vartheta - \mathbf{v}_h), \nabla(\mathbf{u} - \mathbf{u}_h)) \\
 &\quad + (p - p_h, \nabla \cdot (\mathbf{v}_h - \vartheta)) - (\mu - P_h\mu, \nabla \cdot (\mathbf{u}_h - \mathbf{u})), \quad \forall \mathbf{v}_h \in V_h.
 \end{aligned}
 \tag{83}$$

Noting that this holds $\forall \mathbf{F} \in L^2(\Omega)^2$, and that $\mathbf{u} - \mathbf{u}_h + \theta_h \in L^2(\Omega)$, choose \mathbf{F} so that $\|\mathbf{F}\|_0 = 1$ and $(\mathbf{F}, \mathbf{u} - \mathbf{u}_h + \theta_h) = \|\mathbf{u} - \mathbf{u}_h + \theta_h\|_0$. Choosing \mathbf{F} fixes (ϑ, μ) by (77), so (78) gives $\|\vartheta\|_2 + |\mu|_1 \leq C_1 \|\mathbf{F}\|_0 = C_1$. Now choose \mathbf{v}_h as the V_h interpolant of ϑ . Using the interpolant and projection error estimates for $(\vartheta, \mu) \in H^1(\Omega)^2 \times L^2(\Omega)$ results in

$$\|\vartheta - \mathbf{v}_h\|_1 \leq C_2 h \quad \text{and} \quad \|\mu - P_h\mu\|_0 \leq C_3 h.$$

It follows that

$$\|\vartheta - \mathbf{v}_h\|_0 \leq C_2 h \quad \text{and} \quad \|\nabla(\vartheta - \mathbf{v}_h)\|_0 \leq \sqrt{2}C_2 h.$$

Finally, from (83),

$$\begin{aligned}
 \|\mathbf{u} - \mathbf{u}_h + \theta_h\|_0 &\leq \eta\|\vartheta - \mathbf{v}_h\|_0 \|\mathbf{u} - \mathbf{u}_h\|_0 + \nu\|\nabla(\vartheta - \mathbf{v}_h)\|_0 \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 \\
 &\quad + \|p - p_h\|_0 \|\nabla \cdot (\mathbf{v}_h - \vartheta)\|_0 + \|\mu - P_h\mu\|_0 \|\nabla \cdot (\mathbf{u}_h - \mathbf{u})\|_0
 \end{aligned}
 \tag{84}$$

so using the errors proven earlier for the approximation (\mathbf{u}_h, p_h) ,

$$\|\mathbf{u} - \mathbf{u}_h + \nabla\theta_h\|_0 \leq Ch^{1+k} (|\mathbf{u}|_{k+1} + |p|_k),$$

thus proving Theorem 4.1. ■

Table 1. Results for Example 1.

	h	$\ \mathbf{u}_h - \mathbf{u}\ _0$	$\ \mathbf{u}_h - \mathbf{u}\ _1$	$\ p_h - p\ _0$	$ \theta _1$
$\eta = 0, \nu = 1$	$\frac{1}{4}$	6.55e - 1	2.37e + 1	3.98e + 0	4.92e - 2
	$\frac{1}{8}$	8.30e - 2	6.19e + 0	3.06e - 1	8.46e - 3
	$\frac{1}{16}$	1.05e - 2	1.57e + 0	2.18e - 2	6.68e - 4
	$\frac{1}{32}$	1.31e - 3	3.93e - 1	1.50e - 3	4.48e - 5
$\eta = 1, \nu = 1$	$\frac{1}{4}$	6.54e - 1	2.37e + 1	3.99e + 0	4.89e - 2
	$\frac{1}{8}$	8.29e - 2	36.19e + 03	3.07e - 1	8.43e - 3
	$\frac{1}{16}$	1.05e - 2	31.57e + 0	2.18e - 2	6.66e - 4
	$\frac{1}{32}$	1.31e - 3	33.93e - 1	1.50e - 3	4.47e - 5

5. NUMERICAL RESULTS

In this section, numerical results for Problem 3.5 are presented which confirm the convergence rates predicted by Theorem 4.1. The results are given as $L^2(\Omega)$ norms and $H^1(\Omega)$ seminorms of the difference between the finite-element approximation and the exact solution, e.g., $\|p_h - p\|_0$ and $|\mathbf{u}_h - \mathbf{u}|_1$. Also displayed are results for $|\theta|_1$ and $\|\mathbf{u}_h - \mathbf{u}\|_0$, which by Poincare-Friedrichs should satisfy [1]

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C |\mathbf{u}|_1.$$

In each example, the discrete systems generated by the Glowinski-Pironneau algorithm are solved using the Choleski method. The triangular mesh in all cases is constructed so that no triangle has two edges on Γ [13]. As stated in [2], triangulating into corners is not necessary to achieve

Table 2. Results for Example 2.

	h	$\ \mathbf{u}_h - \mathbf{u}\ _0$	$ \mathbf{u}_h - \mathbf{u} _1$	$\ p_h - p\ _0$	$ \theta _1$
Re = 0.01	$\frac{1}{4}$	3.90e - 1	1.14e + 1	1.03e + 3	2.14e - 1
	$\frac{1}{8}$	4.65e - 2	2.94e + 0	1.59e + 2	1.79e - 2
	$\frac{1}{16}$	5.44e - 3	7.41e - 1	2.20e + 1	1.18e - 3
	$\frac{1}{32}$	6.60e - 4	1.86e - 1	2.90e + 02	7.46e - 5
Re = 0.1	$\frac{1}{4}$	3.78e - 1	1.10e + 1	9.99e + 1	2.07e - 1
	$\frac{1}{8}$	4.50e - 2	2.85e + 0	1.54e + 1	1.74e - 2
	$\frac{1}{16}$	5.27e - 3	7.17e - 1	2.13e + 0	1.17e - 3
	$\frac{1}{32}$	6.40e - 4	1.80e - 1	2.80e - 1	8.61e - 5
Re = 1.0	$\frac{1}{4}$	2.77e - 1	8.12e + 0	7.18e + 0	1.49e - 1
	$\frac{1}{8}$	3.31e - 2	2.08e + 0	1.09e + 0	1.33e - 2
	$\frac{1}{16}$	3.97e - 3	5.23e - 1	1.51e - 1	1.31e - 3
	$\frac{1}{32}$	5.21e - 4	1.31e - 1	2.00e - 2	2.39e - 4
Re = 10.0	$\frac{1}{4}$	4.07e - 2	1.21e + 0	7.50e - 2	1.43e - 2
	$\frac{1}{8}$	5.13e - 3	3.07e - 1	1.13e - 2	1.68e - 3
	$\frac{1}{16}$	7.02e - 4	7.69e - 2	1.86e - 3	3.57e - 4
	$\frac{1}{32}$	1.17e - 4	1.92e - 2	3.63e - 4	9.01e - 5
Re = 100.0	$\frac{1}{4}$	1.39e - 2	4.04e - 1	6.82e - 4	4.90e - 4
	$\frac{1}{8}$	1.83e - 3	1.03e - 1	6.54e - 5	4.79e - 5
	$\frac{1}{16}$	2.32e - 4	2.61e - 2	1.24e - 5	3.40e - 6
	$\frac{1}{32}$	2.91e - 5	6.53e - 3	2.91e - 6	2.22e - 7

optimal accuracy. It was observed that for the examples given in this paper, the magnitude (though not the convergence rate) of errors was less when using triangulation into corners.

5.1. Example 1

Results for Example 1 are displayed in Table 1. Here the domain is $[0, 1] \times [0, 1]$ and \mathbf{f} is chosen so that the solution is

$$\begin{aligned} \mathbf{u} &= 256x(x - 1)(2x - 1)(6y^2 - 6y + 1) \mathbf{i} \\ &\quad - 256y(y - 1)(2y - 1)(6x^2 - 6x + 1) \mathbf{j}, \\ p &= \left(x - \frac{1}{2}\right) \left(y - \frac{1}{2}\right). \end{aligned}$$

The errors in $\|\mathbf{u}_h - \mathbf{u}\|_1$ converge at the predicted rates, while the others converge faster than predicted.

5.2. Example 2

The solution used for this example is known as a 2D Kovaszny flow [14]. The domain is $[-1/2, 1/2] \times [-1/2, 1/2]$, with $\eta = 1$, $\nu \in \{1/100, 1/10, 1, 10, 100\}$, and \mathbf{f} selected so that

$$\begin{aligned} \mathbf{u} &= \{-e^{\lambda x} \cos(2\pi y)\} \mathbf{i} + \left\{\frac{\lambda}{2\pi} e^{\lambda x} \sin(2\pi y)\right\} \mathbf{j}, \\ p &= \frac{1 - e^{\lambda x}}{2}, \end{aligned}$$

where $\lambda = \text{Re}/2 - ((\text{Re}/2)^2 + (2\pi)^2)^{1/2}$ and $\text{Re} = 1/\nu$. Convergence results are displayed in Table 2. Results for the case $\eta = 0$ are nearly identical to those for $\eta = 1$.

5.3. Timing Results

Timing comparisons between the Glowinski-Pironneau algorithm, a sparse direct approach, and an iterative algorithm are now presented. Tables 3-5 can be used to compare times, in seconds, to solve (1) using the sparse direct solver in the Aztec package [15], an iterative solver in Aztec, and the Glowinski-Pironneau algorithm using the Choleski method for the Poisson solves. In each case, the exact solution is the Kovaszny flow with $\eta = \nu = 1$. The iterative solver is BiCGStab with a ninth-order least squares polynomial preconditioner, using $1.e - 6$ as the convergence criterion for the relative residual.

The timings in Table 5 are organized to separate that portion of the algorithm which will be repeated for each time step to solve an unsteady problem, in which the right-hand side of (1)

Table 3. Timing in seconds for sparse direct solver.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
Assemble	0.06	0.43	4.56
Factor	0.27	6.06	139.30
Solve	0.01	0.04	0.23

Table 4. Timing in seconds for iterative solver.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
Assemble	0.06	0.43	4.45
Solve	0.10	1.92	11.68

Table 5. Timing in seconds for Glowinski Pironneau algorithm.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
Assemble boundary pressure matrix corresponding to (45), using (42)–(44)	0.28	3.55	49.80
Factor boundary pressure matrix	0.006	0.014	0.076
Assemble and factor systems (39)–(41)	0.08	0.92	13.28
Solve for boundary pressures	0.02	0.14	0.89
– Solve factored form of (39)–(41)			
– Calculate RHS of (45)			
– Solve factored form of (45)			
– Determine $p_{h1}, \mathbf{u}_{h1}, \theta_{h1}$			
– Determine $p_h, \mathbf{u}_h, \theta_h$ using (46)–(48)			

Table 6. Convergence results using sparse LU solver.

h	$\ \mathbf{u}_h - \mathbf{u}\ _0$	$ \mathbf{u}_h - \mathbf{u} _1$	$\ p_h - p\ _0$
$\frac{1}{4}$	2.49e-1	8.47e+0	9.55e+0
$\frac{1}{8}$	3.17e-2	2.10e+0	1.35e+0
$\frac{1}{16}$	3.79e-3	5.24e-1	1.79e-1
$\frac{1}{32}$	4.64e-4	1.31e-1	2.33e-2

changes. This is the case, for example, in the θ -method [4]. It is appropriate, then, to compare the last row of timings in each table. The direct solver, as would be expected, has the lowest time in the solve step. This fact loses relevance as the problem size grows beyond the point at which an LU solver for the velocity-pressure system is feasible. One can infer from a comparison of the solve times for the latter two methods that in a time-dependent context, there will be a threshold number of time steps at which the total time for the iterative method equals that for the Glowinski-Pironneau algorithm, after which the Glowinski-Pironneau algorithm will take less time. More extensive numerical results are needed to confirm this point.

Table 6 displays convergence results for the sparse direct solver from Aztec for the Kovasznay flow with $\eta = \nu = 1$. The entries can be compared with the $Re = 1$ values in Table 2, showing that the accuracy of the algorithms is very similar.

6. CONCLUSIONS AND FUTURE WORK

In this paper, a complete description and analysis of the Glowinski-Pironneau finite-element method for the unsteady Stokes problem has been presented. The next step in this research will be to use the algorithm (or a variant) within a time-dependent viscoelastic flow simulation. Implementation in a parallel setting, for the 3D problem, is a likely direction for this effort.

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