On zero-dimensional continuous images of compact ordered spaces

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Throughout this paper X denotes a compact and zero-dimensional Hausdorff space. We shall be concerned with

Theorem 0.1 *The following assertions are equivalent.*

(A) *X is the continuous image of a compact ordered space.*

(B) *X is the continuous image of a zero-dimensional compact ordered space.*

(C) *X has a T_0-separating cross-free family of clopen sets.*

(D) *X has a T_0-separating non-Archimedian family of clopen subsets.*

(E) *X can be embedded as a strong T-set into a dendron.*

(F) *X can be embedded into a dendron.*

(G) *X has a cross-free closed subbase.*

(H) *There is a continuous multiplication $\cdot$ on X making it a topological semilattice, i. e., associative, commutative, and idempotent, such that $x \cdot y \cdot z \in \{x \cdot y\, ,\, x \cdot z\}$ for all $x, y, z \in X$.*

Although probably never arranged in single file, the above equivalences are more or less known. The implication $G \Rightarrow F$ is due to van Mill and Wattel (cf. [7, theorem 6.6]) and does not depend on X being zero-dimensional (but the general case it is much harder to prove).

$F \Rightarrow A$ follows from the fact that every dendron is a continuous image of a compact ordered space (a result due independently to J. L. Cornette [3] and A. E. Brouwer [2] see also [7, theorem 3.3]).

The implication $A \Rightarrow E$ is due to J. Nikiel [9]. S. Purisch recently observed $E \Rightarrow D$, which he combined with Nikiel's result to obtain $A \Rightarrow D$ (cf. [12]), thus solving a problem posed by van Douwen in [4] and in Boolean algebraic form also by Koppelberg and Monk in [6].

The equivalence of F and H does not come out of the blue either. But it is not clear how to get it directly from L. E. Ward's result in [11].

In this note we offer new proofs of most of the implications according to the following schema.

$$\begin{array}{ccccccccccccccccc} E & \overset{t}{\Longrightarrow} & F & \Longrightarrow & G & \overset{3}{\Longrightarrow} & C & \overset{2}{\Longrightarrow} & D & \overset{5}{\Longrightarrow} & H & \overset{2}{\Longrightarrow} & D & \overset{5}{\Longrightarrow} & E \\ & & & & & & & & \Downarrow & & & & & & \\ & & & & B & \overset{t}{\Longrightarrow} & A & \overset{2}{\Longrightarrow} & B & \overset{4}{\Longrightarrow} & C, & & & & \end{array}$$

where trivial implications are marked by $\overset{t}{\Longrightarrow}$ and $\overset{i}{\Longrightarrow}$ means that the implication is proved in section i.

The two unmarked implications will not be proved here, because we have no simplifications to offer. In [7] the very first thing van Mill and Wattel prove about dendrons is the fact that they have cross-free closed subbases. From this the implication $F \Rightarrow G$ is immediate. The second exception is $D \Rightarrow B$. Its Boolean algebraic reformulation (see section 4 below) is proved very elegantly in [6, theorem 3.1].

The author wishes to thank J. D. MONK for making [8] available to him.

1 Definitions

Before starting we recall the definitions of the less common notions involved in the statement of the theorem.

A family $\mathcal{C}$ of subsets of some set X will be called

- T_0–separating if for all $x \neq y$ in X there is some $c \in \mathcal{C}$ such that $x \in c \Leftrightarrow y \notin c$.
- *cross-free* iff for all $c, d \in \mathcal{C}$
  (cf) $\quad c \subseteq d$ or $d \subseteq c$ or $c \cap d = \emptyset$ or $c \cup d = X$.
- *non-Archimedean* (following [4], the phrase *rank 1 family* is also popular) if for all $c, d \in \mathcal{C}$
  (nA) $\quad c \subseteq d$ or $d \subseteq c$ or $c \cap d = \emptyset$.

With the obvious modification in mind we shall use the phrases 'cross-free' and 'non-Archimedian' also for subfamilies of Boolean algebras.

A *dendron* is a compact connected Hausdorff space D such that for all distinct $x, y \in D$ there is a point $z \in D$ which separates x and y, i.e., $D \setminus \{z\} = U \cup V$ for some U and V that are disjoint, open, and such that $x \in U$, $y \in V$.

Finally, a subset X of a dendron D is called a *strong T-set* if it is closed and each connected component of $D \setminus X$ is homeomorphic with the open unit interval $(0, 1)$. A good reference for dendrons is the already cited survey [7]. To understand the proofs below, no theory is needed however.

2 The easy implications

Proof of $A \Longrightarrow B$. Let $f : K \to X$ be a continuous surjection, where K is an ordered compact space. By ZORN's Lemma we can replace K by a minimal closed subspace K' such that $f(K')$ is still the whole of X. A well-known elementary result tells us that K' is also an ordered space.

Let L be a non-empty connected subset of K'. We want to show that L consists of one point only. But otherwise, it would contain a non-empty, densely ordered closed interval, $[l_1, l_2]$ say. As X is zero-dimensional, the connected set $f([l_1, l_2])$ consists of only one point, hence $f(K' \setminus (l_1, l_2)) = f(K') = X$, contradicting minimality. So K' is totally disconnected, hence zero-dimensional. $\square$

The **Proof of $C \Longrightarrow D$.** starts with a

CLAIM *If $\mathcal{C}$ is cross-free then $\mathcal{C} \cup \{X \setminus c : c \in \mathcal{C}\}$ is also cross-free.*

Indeed, rewriting condition (cf) as

$$c \cap (X \setminus d) = \emptyset \ \text{ or } \ d \cap (X \setminus c) = \emptyset \ \text{ or } \ c \cap d = \emptyset \ \text{ or } \ (X \setminus c) \cap (X \setminus d) = \emptyset$$

it becomes clear that the sets c and d and their complements enter symmetrically, from which the claim follows.

Now let $\mathcal{C}$ be a cross-free T_0-separating family of clopen sets and fix any point $x \in X$. By the claim, $\mathcal{T} = \{c \in \mathcal{C} : x \notin c\} \cup \{X \setminus c : x \in c \in \mathcal{C}\}$ is also cross-free and, as $x \notin t$ for all $t \in \mathcal{T}$, even non-Archimedian. A moment's reflection shows that $\mathcal{T}$ remains T_0 separating. $\square$

Proof of $H \Longrightarrow D$. Let $(X,\cdot)$ be a topological semilattice such that $x \cdot y \cdot z \in \{x \cdot y\,,\, x \cdot z\}$ for all $x, y, z \in X$. We need a simple fact from topological algebra, which the reader probably knows, at least for groups.

Lemma 2.1 (NUMAKURA's Theorem [10]) *Every compact and zero-dimensional semigroup is profinite.*

For semilattices, i.e., commutative and idempotent semigroups, the assertion can be strengthened using the following

CLAIM *For each pair of distinct elements h_1, h_2 of the* finite *semilattice H there is a homomorphism $\varphi : H \to \{0,1\}$ (multplicative) such that $\varphi(h_1) \neq \varphi(h_2)$.*

To see this, assume without loss $h_1 \cdot h_2 \neq h_1$ and put

$$\varphi(h) = \begin{cases} 1, & h_1 \cdot h = h_1 \\ 0, & \text{otherwise} \end{cases}$$

Then a straight-foreward verification (using associativity, commutativity, and idempotency) shows that φ is as desired.

Combining the claim with NUMAKURA's Theorem we conclude that any pair of points of X is separated by a continuous homomorphism $\varphi : X \to \{0,1\}$. In other words, the family $\mathcal{T}$ consisting of all $\varphi^{-1}(1)$ is T_0-separating. It remains to check that $\mathcal{T}$ is non-Archimedian.

Let φ and ψ be continuous homomorphisms $X \to \{0,1\}$. Aiming at a contradiction we assume

$$\varphi^{-1}(1) \not\subseteq \psi^{-1}(1) \quad \text{and} \quad \psi^{-1}(1) \not\subseteq \varphi^{-1}(1) \quad \text{and} \quad \psi^{-1}(1) \cap \varphi^{-1}(1) \neq \emptyset.$$

This yields points $x, y, z \in X$ such that

$$\varphi(y) = 1,\ \psi(y) = 0 \quad \text{and} \quad \psi(z) = 1,\ \varphi(z) = 0 \quad \text{and} \quad \varphi(x) = \psi(x) = 1.$$

Then $\varphi(x \cdot y \cdot z) = \psi(x \cdot y \cdot z) = 0$, but $\varphi(x \cdot y) = 1$ and $\psi(x \cdot z) = 1$, contradicting $x \cdot y \cdot z \in \{x \cdot y\,,\, x \cdot z\}$ $\square$

3 Closed versus clopen

Assume that $\mathcal{S}$ is a cross-free closed subbase of X. We show that $\mathcal{S}$ contains enough clopen elements to separate the points of X, thus establishing the implication $G \Rightarrow C$.

There is no harm in assuming $X \in \mathcal{S}$. But then

CLAIM *the finite unions of pairwise disjoint elements of $\mathcal{S}$ constitute a closed base of X.*

To see this, consider any finite union $S_1 \cup \ldots \cup S_n$ of elements of $\mathcal{S}$. If $S_i \subseteq S_j$ for some $i \neq j$, we may drop S_i. If $S_i \cup S_j = X$ we can replace the whole union by $X \in \mathcal{S}$. Having purged it this way, the union has become disjoint.

Now let $x \neq y$ be given and choose a clopen set A such that $x \in A \not\ni y$. As A is closed, it can be represented as an intersection of basic closed sets. As A is also open, a finite intersection

$$A = \bigcap_{i=1}^{n} (\, S_1^i \,\cup \ldots \cup\, S_{m_i}^i \,)$$

is sufficient (by compactness). By the claim, we can further assume that $S_p^i \cap S_q^i = \emptyset$ for all i and $1 \leq p < q \leq m_i$. Using the distributivity of the set-theoretic operations we rewrite

$$A = \bigcup_f \bigcap_{i=1}^{n} S_{f(i)}^i,$$

where f runs over all mappings $\{1,\ldots,n\} \longrightarrow \{1,\ldots,\max m_i\}$ such that $f(i) \leq m_i$ for all i.

The sets $\bigcap_{i=1}^n S^i_{f(i)}$ are, obviously, closed. They are also pairwise disjoint, for $f \neq g$ means $f(j) \neq g(j)$ for some j, hence

$$\bigcap_{i=1}^{n} S^i_{f(i)} \cap \bigcap_{i=1}^{n} S^i_{g(i)} \subseteq S^j_{f(j)} \cap S^j_{g(j)} = \emptyset.$$

As their union A is open, all $\bigcap_{i=1}^n S^i_{f(i)}$ are clopen and one of them separates x and y.

To simplify notation, we now consider a clopen intersection $B = S_1 \cap \ldots \cap S_n$ of (without loss) pairwise incomparable elements of $\mathcal{S}$, such that $x \in B \not\ni y$ There is no loss in further assuming that $x \in S_1 \not\ni y$. Our aim is to show that S_1 is actually clopen. For $j = 2,\ldots,n$ we must have $S_1 \cup S_j = X$, for $x \in S_1 \cap S_j \neq \emptyset$ and comparability is excluded by assumption. It follows that $X = S_1 \cup (S_2 \cap \ldots \cap S_n)$, hence, denoting $(S_2 \cap \ldots \cap S_n)$ by C,

$$X = S_1 \cup C = [\,S_1 \setminus (S_1 \cap C)] \cup [S_1 \cap C] \cup [\,C \setminus (S_1 \cap C)].$$

Given that $S_1 \cap C = B$ is clopen, this is a decomposition of X into three disjoint closed sets. It follows that they are all clopen and so is

$$S_1 = [\,S_1 \setminus (S_1 \cap C)] \cup [S_1 \cap C],$$

as was to be shown. □

4 Switching to Boolean algebras

In this section we prove the implication $B \Longrightarrow C$ using the language of Boolean algebras.

If $f : K \rightarrow X$ is a continuous surjection, then the assignment $a \mapsto f^{-1}(a)$ defines an embedding $Clop(X) \rightarrow Clop(K)$ of the Boolean algebras consisting of all clopen subsets of X, resp. K, under the set-theoretic operations.

The fact that K is an ordered space translates to $Clop(K)$ being a so-called *interval algebra*. This fact and much background on interval algebras can be found in section 15 of [5].

Each point $x \in X$ gives rise to the ultrafilter $\{a \in Clop(X) : x \in a\}$ of $Clop(X)$. It follows that in order to establish $B \Longrightarrow C$ it is sufficient to prove

Proposition 4.1 *Every subalgebra of an interval algebra contains a cross-free subcollection which separates its ultrafilters.*

Before embarking on that proof we introduce some notation.

Let R be a linear ordering with least element 0. The *interval algebra on* R, to be denoted by $Intalg(R)$, is defined to be the subalgebra of the power-set algebra on R generated by the collection of all half-open intervals $[r, s) = \{t \in R : r \leq t < s\}$. As $Intalg(R)$ is an algebra of sets, we use $\cap, \cup$ and $\setminus$ to denote the Boolean operations and $\emptyset$ resp. R for the zero- resp. unit element.

It is well-known that each non-zero element b of $Intalg(R)$ has a unique representation of the form

$$b = [r_0, r_1) \cup \ldots \cup [r_{2n}, r_{2n+1}),$$

where $n \geq 0$ and $0 \leq r_0 < r_1 < r_2 < \ldots < r_{2n} < r_{2n+1} \leq \infty$, with the obvious meaning of $[r, \infty)$. The elements $r_0, r_2, \ldots, r_{2n}$ will be called the *left end-points* of b, whereas $r_1, r_3, \ldots, r_{2n+1}$ are its *right end-points*. By rel(b) we denote the set $\{r_0, r_1, \ldots, r_{2n}, r_{2n+1}\} \setminus \{0, \infty\}$ (the *relevant points* of b). In particular, for the unit element we have $\mathrm{rel}(R) = \mathrm{rel}([0,\infty)) = \emptyset$. By definition, we put $\mathrm{rel}(\emptyset) = \emptyset$.

We need the following

Observations 4.2 *concerning* $a, b \in Intalg(R)$ *and* $m \in R$.

(1) $\mathrm{rel}(R \setminus b) = \mathrm{rel}(b)$

(2) *If m is a left end-point of a and $m \in b$, then* $\mathrm{rel}(a \setminus b) \subseteq (\mathrm{rel}(a) \cup \mathrm{rel}(b)) \setminus \{m\}$.

(3) *If m is a left end-point of a and $m \notin b$, then* $\mathrm{rel}(a \cap b) \subseteq (\mathrm{rel}(a) \cup \mathrm{rel}(b)) \setminus \{m\}$.

Verifications are very simple and, therefore, omitted.

Now we prove the proposition. Let R be as before and let B be a subalgebra of $Intalg(R)$.

Claim *We can find a set $M \subseteq R \setminus \{0\}$ such that*

$(m1)$ $\mathrm{rel}(b) \cap M \neq \emptyset$ *for all $b \in B \setminus \{\emptyset, R\}$, and*

$(m2)$ *for each $m \in M$ there exists exactly one $b \in B$ such that $m \in B$ and* $\mathrm{rel}(b) \cap M = \{m\}$.

To see this, consider the collection $\mathcal{N}$ of all $N \subseteq R \setminus \{0\}$ for which $(m1)$ holds. Containing $R \setminus \{0\}$, the collection is not empty. Moreover, $\mathcal{N}$ is closed under intersections of descending chains (as $\mathrm{rel}(b)$ is finite for each b). So, by Zorn's Lemma, there is a minimal element, M say, in $\mathcal{N}$.

Automatically M satisfies $(m1)$. To see that it also satisfies $(m2)$, we fix $m \in M$. By minimality, $M \setminus \{m\} \notin \mathcal{N}$. So $\mathrm{rel}(b) \cap (M \setminus \{m\}) = \emptyset$, hence $\mathrm{rel}(b) \cap M = \{m\}$ for some $b \in B$. Switching to $R \setminus b$ if necessary, we can additionally assume that m is a left end-point of b, i.e. $m \in b$. So, b is as desired in (2). To see that b is unique, consider a second element c with the same properties. By 4.2(2), m is not an end-point of either $b \setminus c$ or $c \setminus b$. It follows that $(\mathrm{rel}(b \setminus c) \cup \mathrm{rel}(c \setminus b)) \cap M = \emptyset$, which, by (1), implies $b \setminus c = c \setminus b = \emptyset$, i.e. $b = c$.

We now fix M as in the claim and denote by c_m the unique element of B such that $m \in c_m$ and $\mathrm{rel}(c_m) \cap M = \{m\}$. Notice that, by $(m2)$ and 4.2(1),

$(m3)$ $R \setminus c_m$ is the unique element of B such that $m \notin R \setminus c_m$ and $\mathrm{rel}(R \setminus c_m) \cap M = \{m\}$.

In order to see that $C_M = \{c_m : m \in M\}$ is cross-free, consider $m \neq n \in M$.

If $m \in c_n$, then, by 4.2(2), $\mathrm{rel}(c_m \setminus c_n) \cap M \subseteq \{n\}$. If $\mathrm{rel}(c_m \setminus c_n) \cap M = \emptyset$, then, by $(m1)$, $c_m \subseteq c_n$.

If $\mathrm{rel}(c_m \setminus c_n) \cap M = \{n\}$, then, from $n \notin c_m \setminus c_n$ we get $c_m \setminus c_n = R \setminus c_n$, by $(m3)$. It follows that $c_m \cup c_n = R$.

If $n \in c_m$, we argue symmetrically.

If $m \notin c_n$ and $n \notin c_m$, then, by 4.2(3), neither m nor n is in $\mathrm{rel}(c_m \cap c_n)$, so $c_m \cap c_n = \emptyset$, by $(m1)$.

Finally, we establish that C_M separates the ultrafilters of B. Aiming at a contradiction we assume that $p \neq q$ are ultrafilters of B that can not be separated by an element of C_M. Among the elements of B that separate p and q we choose some b such that the (finite) set $\mathrm{rel}(b) \cap M$ is minimal. Take $m \in \mathrm{rel}(b) \cap M$ (which is possible, by $(m1)$). Without loss we can assume that m is a left end-point of b (otherwise we replace b by $R \setminus b$) and that $b \in p$, $b \notin q$ (otherwise we interchange p and q). As c_m does not separate p and q, we either have $c_m \in p$, $c_m \in q$ or $c_m \notin p$, $c_m \notin q$.

In the first case we put $d = c_m \setminus b$. Then $d \notin p$, $d \in q$ and, by 4.2(2),

$$\mathrm{rel}(d) \cap M \subseteq [(\mathrm{rel}(c_m) \cup \mathrm{rel}(b)) \setminus \{m\}] \cap M \subseteq (\mathrm{rel}(b) \cap M) \setminus \{m\},$$

which contradicts the choice of b. In the second case we get the same contradiction using $d = b \setminus c_m$.
□

We close this section with a few further remarks adressed to Boolean algebraists.

The class of pseudo-tree Boolean algebras was introduced by Monk. Many information about these algebras can be found in [6]. The following definition is equivalent (cf. [6, theorem 2.3]) to the original one, but more convenient for our discussion.

Definition 4.3 A Boolean algebra is called a *pseudo-tree algebra* if it has a non-Archimedian set of generators.

The name can be motivated by the fact that each non-Archimedian subset T of a Boolean algebra is a (downward growing) pseudo-tree under the Boolean partial order, i.e., such that for each $t \in T$ the subset $\{s \in T : s \geq t\}$ is a chain. (To make T a(n upside down) tree, these chains have to be (reversely) well-ordered.)

Recalling the implication $C \Rightarrow D$ from section 2 and the fact that a subset of $Clop(X)$ is T_0 separating iff it generates $Clop(X)$ as a Boolean algebra, we can reformulate the above proposition as follows.

Proposition 4.4 *Every subalgebra of an interval algebra is a pseudo-tree algebra.*

In this form the result answers a question of Koppelberg and Monk in [6]. For the special case of superatomic Boolean algebras (corresponding to scattered spaces) it was earlier obtained by Bonnet [1] with a stronger conclusion.

The converse of 4.4 is also true and this yields the implication $D \Longrightarrow B$ of our main theorem. As we mentioned already in the introduction, we can not improve upon the very nice demonstration given in [6, theorem 3.1]. The fact itself was known to van Douwen (cf. [4, theorem 1.3]) and also proved by M. Bekkali in his thesis.

¿From 4.4 and its converse we immediately get

Corollary 4.5 *Every subalgebra of a pseudo-tree algebra is also a pseudo-tree algebra.*

5 The embedding in a dendron

Here we prove the implication $D \Rightarrow E$. A few additional remarks will then yield $D \Rightarrow H$.

Let $\mathcal{T}$ be as in D, i.e. a family of clopen subsets of X such that

($t1$) for distinct $x, y \in X$ there exists $t \in \mathcal{T}$ such that $x \in t$ iff $y \notin t$, and

($t2$) $t \subseteq s$ or $s \subseteq t$ or $s \cap t = \emptyset$ for all $s, t \in \mathcal{T}$

Claim *We can assume that $\mathcal{T}$ has two additional properties, namely*

($t3$) $\emptyset, X \notin \mathcal{T}$ *and*

($t4$) $t_1 \triangle t_2 \triangle \ldots \triangle t_n \neq \emptyset$ *(symmetric difference) for all pairwise distinct* $t_1, t_2, \ldots t_n \in \mathcal{T} \cup \{X\}$.

There is no problem with ($t3$), of course. If ($t4$) is not satisfied by our first choice for $\mathcal{T}$, we replace it by a subset, $\mathcal{T}'$ say, which is maximal with respect to having properties ($t3$) and ($t4$). The existence follows from Zorn's Lemma again. To see that $\mathcal{T}'$ still satisfies ($t1$), we consider $x \neq y$ in X. There is some $t \in \mathcal{T}$ separating x and y. If $t \in \mathcal{T}'$, we are done. Otherwise, by maximality,

$$t \triangle t'_1 \triangle \ldots \triangle t'_n = \emptyset \quad \text{hence} \quad t = t'_1 \triangle \ldots \triangle t'_n$$

for suitable pairwise distinct $t'_i \in \mathcal{T}' \cup \{X\}$. It follows that one of the $t'_i \in \mathcal{T}'$ separates x and y, for, otherwise t would not.

Properties ($t3$) and ($t4$) have the consequence that

($t5$) for $t_1, \ldots, t_n, s_1, \ldots, s_m \in \mathcal{T} \cup \{X\}$ the inclusion $t_1 \cap \ldots \cap t_n \subseteq s_1 \cup \ldots \cup s_m$ is possible only if either $t_i \cap t_j = \emptyset$ or $t_i \subseteq s_j$ for some i, j.

To see this, assume that $t_i \cap t_j \neq \emptyset$ for all i, j. Then, by ($t2$), the t_i are pairwise comparable and their intersection reduces to one term, t_1 say. Among the elements $s_1, \ldots, s_m$ we first drop those that do not intersect t_1 and then those that are not maximal. The remaining elements, $s_1, \ldots, s_k$ say, still cover t_1 and are pairwise disjoint, by ($t2$). Therefore, $s_1 \cup \ldots \cup s_k = s_1 \triangle \ldots \triangle s_k$.

Assuming $t_1 \not\subseteq s_j$ for all j, we get $t_1 \supseteq s_j$ for all j. Consequently, $t_1 \supseteq s_1 \cup \ldots \cup s_k = s_1 \triangle \ldots \triangle s_k \supseteq t_1$, contradicting $(t4)$.

After these preparations we describe the required dendron. It will be called D and is the subspace of the TYCHONOFF cube $[0,1]^{\mathcal{T}}$ consiststing of all $\varphi : \mathcal{T} \to [0,1]$ such that

$(d1)$ $s \subseteq t$ implies $\varphi(s) \leq \varphi(t)$,

$(d2)$ $s \cap t = \emptyset$ implies that either $\varphi(s) = 0$ or $\varphi(t) = 0$, and

$(d3)$ there is at most one value $t \in \mathcal{T}$ such that $\varphi(t) \notin \{0,1\}$.

The desired properties will be established in a series of claims. Some routine verifications will be skipped.

(1) D is closed in the cube, hence compact.

(2) X is homeomorphic with the subspace Y of D consisting of all those $\varphi : \mathcal{T} \to \{0,1\}$ which satisfy $(d1)$ and $(d2)$, i.e., those elements of D that take only the values 0 and 1.

To see this put $F(x)(t) = \begin{cases} 1, & x \in t \\ 0, & x \notin t \end{cases}$ and verify that $F : X \to Y$ is the desired homeomorphism. Condition $(t1)$ yields injectivity, whereas $(t5)$ is needed to get surjectivity.

Next we need names for some elements of D. For $t \in \mathcal{T}$ and $\theta \in [0,1]$ we define $\varphi_{t,\theta}$ by setting

$$\varphi_{t,\theta}(s) = \begin{cases} 1, & t \subset s \text{ (strong inclusion)} \\ \theta, & s = t \\ 0, & t \not\subseteq s. \end{cases}$$

and put $J_t = \{\varphi_{t,\theta} : \theta \in (0,1)\}$.

(3) $\varphi_{t,\theta} \in D$

(4) If $\varphi \in D$ and $\varphi(t) = \theta \in (0,1)$, then $\varphi = \varphi_{t,\theta}$.

It is clear that $\varphi(t) = \varphi_{t,\theta}(t)$. For $s \neq t$, both functions take only values in $\{0,1\}$.
Case 1. $t \subset s$, i.e., $\varphi_{t,\theta}(s) = 1$.
Then, by $(d1)$, $0 < \theta = \varphi(t) \leq \varphi(s)$, hence $\varphi(s) = 1$.
Case 2. $t \not\subseteq s$, i.e., $\varphi_{t,\theta}(s) = 0$.
If $s \cap t = \emptyset$, then $\varphi(s) = 0$, for, otherwise, by $(d2)$, $0 = \varphi(t) = \theta > 0$.
If $s \subset t$, then $\varphi(s) \leq \varphi(t) = \theta < 1$, which again implies $\varphi(s) = 0$.

As an immediate consequence of (4) we get

(5) $D \setminus Y = \bigcup_{t \in \mathcal{T}} J_t$.

Given that J_t is, obviously, homeomorphic to $(0,1)$, we get the 'strog T-set' part of E. In order to prove the connectedness of D we need two preparatory claims.

(6) Every non-empty clopen subset of D intersects Y.

Indeed, let $\emptyset \neq A \subseteq D$ be clopen and take $\varphi \in A$. If $\varphi \in Y$, we are done. Otherwise, by (5), $\varphi = \varphi_{t,\theta} \in J_t$ for some t and θ. As J_t is connected, the clopen set A cannot properly intersect its closure $\overline{J_t}$. So $A \supseteq \overline{J_t} \ni \varphi_{t,0} \in Y$.

Let $\zeta \in Y$ denote the mapping $\mathcal{T} \to \{0,1\}$ constantly taking the value zero.

(7) Every non-empty clopen subset of Y which does not contain ζ can be written as a finite union of sets of the form $\{\varphi \in Y : \varphi(t) = 1;\ \varphi(s_1) = \ldots = \varphi(s_m) = 0\}$, where $t \not\subseteq s_i$ (and $m = 0$ is allowed).

Indeed, by $(d1)$ and $(d2)$ the a priori possible more general form

$$\{\varphi \in Y : \varphi(t_1) = \ldots = \varphi(t_n) = 1; \;\; \varphi(s_1) = \ldots = \varphi(s_m) = 0\}$$

reduces to the given one. At least one t must be present because of ζ.

(8) D is connected.

Assume not. Then there is a non-empty clopen subset A of D, which does not contain ζ. By (6), its intersection with Y is non-empty. By (7), we get

$$A \cap Y = \bigcup_{i=1}^{n} \{\varphi \in Y : \varphi(t_i) = 1; \; \varphi(s_1^i) = \ldots = \varphi(s_{m_i}^i) = 0\}.$$

Let t be maximal among $t_1, \ldots, t_n$. Then $\varphi_{t,1} \in A \cap Y$ but $\varphi_{t,0} \notin A \cap Y$. It follows that A splits the connected set J_t, which is impossible.

(9) D is a dendron

Let α and β be distinct points in D. Without loss, we can assume $\alpha(t) < \beta(t)$ for some $t \in \mathcal{T}$. Choosing θ such that $\alpha(t) < \theta < \beta(t)$, we obtain, by (4),

$$D \setminus \{\varphi_{t,\theta}\} = \{\varphi \in D : \varphi(t) < \theta\} \cup \{\varphi \in D : \varphi(t) > \theta\},$$

which is the desired decomposition. The implication $D \Rightarrow E$ is proved. □

To **prove** $D \Rightarrow H$ it will be sufficient to introduce the required semilattice structure on the space $Y \subseteq \{0,1\}^{\mathcal{T}}$ constructed above. The operation $\cdot$ of pointwise multiplication on the cube $\{0,1\}^{\mathcal{T}}$ is, obviously, associative, commutative, and idempotent.

Moreover, Y is closed under $\cdot$. Indeed, if $\varphi, \psi : \mathcal{T} \to \{0,1\}$ are in Y, i.e., satisfy $(d1)$ and $(d2)$, then the same is true of their pointwise product $\varphi \cdot \psi$.

It remains to verify that for all φ, ψ, χ satisfying $(d1)$ and $(d2)$ either $\varphi \cdot \psi \cdot \chi = \varphi \cdot \psi$ or $\varphi \cdot \psi \cdot \chi = \varphi \cdot \chi$. If that were not true, we could find s and t in $\mathcal{T}$ such that

$$\varphi(s) = 1, \; \psi(s) = 1, \; \chi(s) = 0, \quad \text{and} \quad \varphi(t) = 1, \; \psi(t) = 0, \; \chi(t) = 1.$$

But then neither $s \cap t = \emptyset$ (because of φ) nor $s \subseteq t$ (because of ψ) nor $t \subseteq s$ (because of χ). This contradicts $\mathcal{T}$ being non-Archimedian. □

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