ON THE DEPENDENCE OF ASYMPTOTICS OF S-NUMBERS OF FRACTIONAL INTEGRATION OPERATORS ON WEIGHT FUNCTIONS

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Abstract

We show that the singular values σ_m of the fractional integration operator of order $\alpha > 0$ acting from weighted L_2 spaces of functions defined on the interval $(-1, 1)$ or $(0, \infty)$ may, in both cases, have asymptotics $m^{-\alpha}$ or $m^{-\alpha/2}$, as $m \to \infty$, depending on the specific choice of weight functions. Some questions of singular value decompositions are also investigated for Riesz potential operators in weighted L_2 spaces of functions defined on $\mathbf{R}^n$.

KEY WORDS: Abel integral operator, Riesz potential operator, singular value decomposition, singular value asymptotics, weighted spaces of square-integrable functions

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1. Introduction

In the recent paper [7] there was discovered the following effect : *singular values of the Riemann-Liouville fractional integration operator* I_0^α : $L_2(R^1_+, e^{-x}) \longrightarrow L_2(R^1_+, x^\alpha e^{-x})$ *have the asymptotics*

$$\sigma_m \sim m^{-\alpha/2}$$

as $m \to \infty$ *, not* $\sigma_m \sim m^{-\alpha}$ *as one could expect from the smoothing properties of the operator* I_0^α *.*

We show that this effect is not a result of non-compactness of R^1_+ : we may have the same effect on a finite interval, and on the contrary, we may have $\sigma_m \sim m^{-\alpha}$ on R^1_+, these effects depending on the special choice of weight functions.

We discuss some arising questions and give some extensions to the multidimensional case, for the Riesz potential operator in weighted $L^2(R^n)$-spaces. In case $n = 1$ we explicitly construct singular systems for the potential type operators, both over R^1 and over a finite interval. We give also some conclusions for the Liouville fractional integration operators over the whole real line as a corollary of our results for the potential type operator.

It should be noted that there exist many investigations on the asymptotics of singular numbers of integral operators. We refer for example to the papers [1] - [3], where some powerful tools were developed for finding the asymptotics of singular numbers of certain pseudo-differential operators of general types. See also [14] where one can find results for the asymptotics in case of fractional integration operators of order $\alpha > 1/2$ with a certain type of a weight function, where a dependence of the asymptotics on a weight function is given in certain terms. We would like to emphasize that because of the applied aspects of the considered operators we prefer to give the direct construction of a sequence of s-numbers, not only its asymptotics, together with the constructive realization of the "singular value decomposition". Besides, even in the case of asymptotics, our weighted considerations are not covered by the known results, up to our knowledge.

2. Preliminaries

1). Fractional integral operators.

We use both the left- and right-sided Riemann-Liouville fractional integrals

$$I^{\alpha}_{a+}u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t)dt \ , \quad x > a \ , \quad \alpha > 0 \ , \tag{1}$$

$$I^{\alpha}_{b-}u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} u(t)dt \ , \quad x < b \ , \alpha > 0 \ , \tag{2}$$

and deal mainly with the cases $a = 0$ or $a = -1$, and $b = 1$.

The corresponding Liouville fractional integrals on the whole real line R^1 will be denoted as

$$I^{\alpha}_{+}u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} u(t)dt \ , \quad x \in R^1 \ , \quad \alpha > 0 \ , \tag{3}$$

$$I^{\alpha}_{-}u(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} u(t)dt \ , \quad x \in R^1 \ , \alpha > 0 \ , \tag{4}$$

The one-dimensional (fractional) Riesz potential has the form

$$I^{\alpha}u = \frac{1}{2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}} \int_{-\infty}^{\infty} \frac{u(y) \ \ dy}{|x-y|^{1-\alpha}} \ , \ x \in R^1, \alpha > 0, \alpha \neq 1, 3, 5, ... \ . \tag{5}$$

We shall also treat its modification for the finite interval

$$A^{\alpha}u = \int_{-1}^{1} \frac{u(y) \ \ dy}{|x-y|^{1-\alpha}} \ , \ |x| < 1. \tag{6}$$

The multidimensional version of the operator (5) is

$$I^{\alpha}u \ = \ \frac{1}{\gamma_n(\alpha)} \int_{R^n} \frac{u(y)dy}{|x-y|^{n-\alpha}} \ , \quad x \in R^n, \tag{7}$$

where $\alpha > 0, \alpha \neq n, n+2, n+4, ...$ and

$$\gamma_n(\alpha) \ = \ \frac{2^{\alpha}\pi^{n/2}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$$

is the normalizing constant for the Riesz potential, see [19] , section 25.2.

We shall consider the above operators within the frameworks of weighted L^2-spaces

$$L^2(\Omega;\rho(x)) = \left\{ f(x) : \int_\Omega |f(x)|^2 \rho(x) dx < \infty \right\}$$

over the corresponding set Ω with some special types of weight functions $\rho(x)$.

2). On boundedness of fractional integration operators .

Lemma 1 . *Let the real numbers μ, ν and ν_1 satisfy the conditions $\mu < 1$ and $\nu_1 = \nu - 2\alpha$ if $\nu - 2\alpha > -1$ and $\nu_1 > -1$ if $\nu - 2\alpha \leq -1$. Then the operator I^α_{0+} is bounded from $L^2([0,1]; x^\mu (1-x)^\nu)$ into $L^2([0,1]; x^{\mu-2\alpha}(1-x)^{\nu_1})$.*

Lemma 2. *Let the real numbers μ, μ_1 and ν satisfy the conditions $\nu < 1$ and $\mu_1 = \mu - 2\alpha$ if $\mu - 2\alpha > -1$ and $\mu_1 > -1$ if $\mu - 2\alpha \leq -1$. Then the operator I^α_{1-} is bounded from $L^2([-1,1]; (1+x)^\mu (1-x)^\nu)$ into $L^2([-1,1]; (1+x)^{\mu_1}(1-x)^{\nu-2\alpha})$.*

Lemma 3. *Let the real numbers μ, ν and ν_1 satisfy the conditions $\mu < 1$ and $\nu_1 = \nu$ if $\mu + \nu < 1$ and $\nu_1 > 1 - \mu$ if $\mu + \nu \geq 1$. Then the operator I^α_{0+} is bounded from $L^2\left(R^1_+; x^\mu (1+x)^\nu\right)$ into $L^2\left(R^1_+; x^{\mu-2\alpha}(1-x)^{\nu_1}\right)$.*

Lemma 4. *The operators (3)-(5) are bounded from $L^2(R^1; (1+x^2)^{\frac{\mu}{2}})$ into $L^2(R^1; (1+x^2)^{\frac{\nu}{2}})$ if $\mu > 2\alpha - 1$ and $\nu < \min(\mu, 1) - 2\alpha$.*

Lemmas 1 - 4 are particular cases of more general results on boundedness of operators of fractional integration within the framework of L^p-spaces with power-type weights, see [19], Theorems 3.10 and 5.5. We also note that Lemma 2 can be obtained from Lemma 1 by an obvious linear change of variables, while Lemma 3 can be derived from Lemma 2 by means of the connection (23)-(24) given below.

3). On mapping between the spaces on $[-1,1]$ and R^1_+.

Obviously, the involutive change of variables

$$\frac{1-x}{1+x} = y \ , \quad \frac{1-y}{1+y} = x \tag{8}$$

maps the half-axis R^1_+ onto $[-1,1]$ and *vice versa.*

We introduce the operator

$$M_\alpha f(x) \ = \ f^*(x) \ = \ \gamma(1+x)^{-1-\alpha} f\left(\frac{1-x}{1+x}\right), \tag{9}$$

associated with the order α of fractional integration. The normalizing factor

$$\gamma = 2^{\frac{\mu+\nu+1}{2}}$$

is chosen to achieve isometry between weighted spaces considered in Lemma 4 below.

We note that the expression for the operator inverse to M_α coincides with the expression for M_α up to a constant numerical factor :

$$M_\alpha^{-1} f(x) = 2^{\alpha-\mu-\nu} M_\alpha f(x) \tag{10}$$

Lemma 5. *Let μ, ν and α be arbitrary real numbers. Then the operator M_α maps the space $L^2\left([-1,1];(1+y)^\mu(1-y)^\nu\right)$ onto $L^2\left(R^1_+; x^\nu(1+x)^{2\alpha-\mu-\nu}\right)$ preserving the scalar product:*

$$\int_{-1}^{1} f_1(y) f_2(y)(1+y)^\mu (1-y)^\nu dy = \int_{-\infty}^{\infty} f_1^*(x) f_2^*(x) x^\nu (1+x)^{2\alpha-\mu-\nu} dx \ ,$$

Remark 1. *In view of (1)-(3) we also have the relation*

$$\int_{-\infty}^{\infty} f_1(x) f_2(x) x^\nu (1+x)^{2\alpha-\mu-\nu} dx = 2^{2(\alpha-\mu-\nu)} \int_{-1}^{1} f_1^*(y) f_2^*(y)(1+y)^\mu (1-y)^\nu dy$$

4). Singular numbers.

We recall the essentials of the theory of singular numbers of compact linear operators (see details in [4]). Let U and V be infinite-dimensional Hilbert spaces. If $A : U \to V$ is an injective compact linear operator, then there exist

(i) an orthonormal basis $\{u_j\}_{j\in N_0}$ in U ;

(ii) an orthonormal system $\{v_j\}_{j\in N_0}$ in V ;

(iii) a nonincreasing sequence $\{\sigma_j\}_{j\in N_0}$ of positive numbers with limit 0 as $j \to \infty$, such that

$$A u_j = \sigma_j v_j, \quad j \in N_0.$$

The numbers σ_j are known as singular numbers or s-numbers of the operator A and the system $\{\sigma_j, u_j, v_j\}_{j\in N_0}$ is called a singular system for the operator A (in view of what is discussed in the paper, it is appropriate to refer to $\{\sigma_j, u_j, v_j\}_{j\in N_0}$ as a singular system not for the operator A only, but for the operator A and a given pair U and V of the spaces involved.

Remark 2. We note that for any operator A the setting

$$A \ : \ L^2(\Omega; \varphi(x)) \to L^2(\Omega; \psi(x))$$

with a singular system $\{\sigma_m, u_m, v_m\}$ is equivalent to the setting

$$A_1 = \psi^{\frac{1}{2}} A \varphi^{-\frac{1}{2}} : L^2(\Omega) \to L^2(\Omega)$$

with the system $\{\sigma_m, \varphi^{\frac{1}{2}} u_m, \psi^{\frac{1}{2}} v_m\}$.

Proposition 1 ([4] , p.27) . For any bounded linear operator $B : U \to U$ or $B : V \to V$,

$$\sigma_m(AB) \le \|B\|_U \ \sigma_m(A),$$
$$\sigma_m(BA) \le \|B\|_V \ \sigma_m(A),$$

respectively.

In the following theorems , obtained in [7], , $C_n^\lambda(x), P_n^{\alpha,\beta}(x)$ and $L_n^{(\beta)}(x)$ are the standard notations, see e.g. [8] , for the Gegenbauer, Jacobi and Laguerre polynomials correspondingly and

$$a_m = \left[\frac{2^{2\lambda-1}(m+\lambda)m!}{\pi\Gamma(m+2\lambda)}\right]^{1/2} \Gamma(\lambda)\ ,\ \ b_m = \left[\frac{2^{1-2\lambda}(m+\lambda)m!\Gamma(m+2\lambda)}{\Gamma(m+\lambda-\alpha+1/2)\Gamma(m+\lambda+\alpha+1/2)}\right]^{1/2},$$

$$A_m = \left(\frac{m!}{\Gamma(\beta+m+1)}\right)^{1/2}\ ,\ \ B_m = \left(\frac{\Gamma(m+\beta+1)}{\Gamma(m+\alpha+\beta+1)}\right)^{1/2}$$

Theorem A [7] . *Let $\lambda > \alpha - 1/2, \alpha > 0, \lambda \ne 0$. Then the operator*

$$I_{-1+}^\alpha \ :\ \ L^2\left([-1,1]; \left(\frac{1-x}{1+x}\right)^{\lambda-1/2}\right) \to L^2\left([-1,1]; \left(\frac{1-x}{1+x}\right)^{\lambda-1/2} (1-x^2)^{-\alpha}\right) \tag{11}$$

has the following singular system:

$$\sigma_m = \left[\frac{\Gamma(m+\lambda-\alpha+1/2)}{\Gamma(m+\lambda+\alpha+1/2)}\right]^{1/2} \sim m^{-\alpha}\ , \tag{12}$$

$$u_m(x) = a_m(1+x)^{\lambda-1/2} C_m^\lambda(x)\ ,\ \ v_m(x) = b_m(1+x)^{\alpha+\lambda-1/2} P_m^{(\lambda-\alpha-1/2, \lambda+\alpha-1/2)}(x)\ . \tag{13}$$

Theorem B [7] . *Let $\beta > -1$ and $\alpha > 0$. Then the operator*

$$I_{0+}^\alpha : L^2(R_+^1; x^{-\beta}e^{-x}) \to L^2(R_+^1; x^{-\alpha-\beta}e^{-x}) \tag{14}$$

has the following singular system

$$\sigma_m = \left[\frac{\Gamma(m+\beta+1)}{\Gamma(m+\alpha+\beta+1)}\right]^{1/2} \sim m^{-\alpha/2}, \tag{15}$$

$$u_m(x) = A_m x^\beta L_m^{(\beta)}(x) \ , \quad v_m(x) = B_m x^{\alpha+\beta} L_m^{(\alpha+\beta)}(x) \ . \tag{16}$$

As a corollary of Theorem A we also have the following theorem for the adjoint operator

$$I_{1-}^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 (t-x)^{\alpha-1} u(t) dt \ , \quad x < 1 \ .$$

Theorem A′ . *Let* $\lambda > \alpha - 1/2, \alpha > 0, \lambda \neq 0$. *Then the operator*

$$I_{1-}^\alpha \ : \quad L^2\left([-1,1]; \left(\frac{1-x}{1+x}\right)^{1/2-\lambda}\right) \to L^2\left([-1,1]; \left(\frac{1-x}{1+x}\right)^{1/2-\lambda} (1-x^2)^{-\alpha}\right) \tag{17}$$

has the following singular system:

$$\sigma_m = \left[\frac{\Gamma(m+\lambda-\alpha+1/2)}{\Gamma(m+\lambda+\alpha+1/2)}\right]^{1/2} \sim \ m^{-\alpha} \ , \tag{18}$$

$$u_m(x) = a_m (1-x)^{\lambda-1/2} C_m^\lambda(x) \ , \quad v_m(x) = b_m (1-x)^{\alpha+\lambda-1/2} P_m^{(\lambda-\alpha-1/2, \lambda-\alpha-1/2)}(x) \ . \tag{19}$$

To derive Theorem A′ from Theorem A it is sufficient to remark that

$$I_{1-}^\alpha = Q I_{-1+}^\alpha Q \tag{20}$$

where $Qu(x) = u(-x)$ and note that

$$C_m^\lambda(-x) = (-1)^m C_m^\lambda(x) \ , \quad P_m^{(\alpha,\beta)}(-x) = (-1)^m P_m^{(\alpha,\beta)}(x)$$

5). Stereographic projection.

We consider the Euclidean space R^n embedded into R^{n+1} , identifying R^n with the hyperplane $\{x \in R^{n+1} : x_{n+1} = 0\}$. The change of variables in R^{n+1} :

$$\xi = s(x) = \{s_1(x), ..., s_{n+1}(x)\},$$
$$s_k(x) = \frac{2x_k}{1+|x|^2}, \ k = 1, ..., n, \quad s_{n+1}(x) = \frac{|x|^2 - 1}{|x|^2 + 1}, \tag{21}$$

where $x = (x_1, ..., x_{n+1}) \in R^{n+1}, |x| = (x_1^2 + ... + x_{n+1}^2)^{1/2}$, generates the mapping known as the stereographic projection. It maps the subspace $\dot{R}^n$

one-to-one onto the unit sphere $S^n \subset R^{n+1}$, $S^n = \{\xi \in R^{n+1} : \xi_1^2 + ... + \xi_{n+1}^2 = 1\}$, where by $\dot{R}^n$ we denote the complementation of the Euclidean space R^n by a single infinite point. It is known [11] that the following relations hold

$$|\xi - \sigma| \; = \; \frac{2|x-y|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}}, \tag{22}$$

$$d\sigma = \frac{2^n dy}{(1+|y|^2)^n}, \tag{23}$$

where $\xi = s(x)$, $\sigma = s(y)$, $x, y \in R^{n+1}$.

6). Spherical harmonics.

We refer e.g. to [12],[20] for harmonic analysis on S^n and recall only some notations and the Funk-Hekke formula . By $Y_m(\sigma), \sigma \in S^n$, we denote an arbitrary spherical harmonic of order m , that is, the restriction of any homogeneous harmonic polynomial in R^{n+1} onto S^n. The space H_m of all such harmonics has a dimension

$$d_{n+1}(m) \; = \; \frac{n+2m-1}{n+m-1} \left(\begin{array}{c} m+n-1 \\ m \end{array} \right). \tag{24}$$

Let$\{Y_{m\mu}\}_{\mu=1,...,d_{n+1}(m)}$ be an orthonormal basis in H_m . It is known ([12],[20]) that the sequence

$$\{Y_{m\mu}\}_{\mu=1,...,d_{n+1}(m);m\in\mathbf{N_0}}$$

is a basis in $L^2(S^n)$.

The formula

$$\int_{S^n} k(\xi\sigma) Y_m(\sigma) d\sigma \; = \; \lambda_m Y_m(\xi) \; , \; \xi \in S^n, \tag{25}$$

known as the Funk-Hekke formula [9], states that an arbitrary spherical harmonic $Y_m(\xi)$ is an eigen-function for any operator K , defined as

$$Ku \; = \; \int_{S^n} k(\xi\sigma) u(\sigma) d\sigma \; , \xi \in S^n, \tag{26}$$

with the kernel depending only on the scalar product $\xi\sigma$. In (25)

$$\lambda_m = \frac{|S^{n-1}|}{\left(\begin{array}{c} m+n-2 \\ m \end{array} \right)} \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} C_m^\lambda(t) k(t) dt \tag{27}$$

where $\lambda = \frac{n-1}{2}$, $C_m^\lambda(t)$ is the Gegenbauer polynomial and $|S^{n-1}|$ is the surface area of the unit sphere in R^n , $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$.

3. Statements and comments

Theorems 1 and 2 below show that the effect of $\sigma_m \sim m^{-\alpha/2}$ is not a result of the consideration of the non-compact set R^1_+ instead of a compact interval. We show that the same operator I^α_{0+} has different asymptotics of s-numbers in two different settings on a finite interval:

$$I_0^\alpha \ : \ L^2([0,1];\rho_1(x)) \to L^2([0,1];r_1(x)) \ , \quad \sigma_m \sim m^{-\alpha} \ , \tag{28}$$

$$I_0^\alpha \ : \ L^2([0,1];\rho_2(x)) \to L^2([0,1];r_2(x)) \ , \quad \sigma_m \sim m^{-\alpha/2} \ . \tag{29}$$

The case (28) is contained in Theorem A , while the case (29) is covered by Theorem 2 below.

Similarly, we may have both situations on the half-axis R^1_+ :

$$I_0^\alpha \ : \ L^2(R^1_+;\varphi_1(x)) \to L^2(R^1_+;\psi_1(x)) \ , \quad \sigma_m \sim m^{-\alpha} \ , \tag{30}$$

$$I_0^\alpha \ : \ L^2(R^1_+;\varphi_2(x)) \to L^2(R^1_+;\psi_2(x)) \ , \quad \sigma_m \sim m^{-\alpha/2}, \tag{31}$$

the case (31) being contained in Theorem B , while the case (30) is covered by Theorem 1 below.

Theorem 1 . *The asymptotics $\sigma_m \sim m^{-\alpha}$ takes place for the operator $I^\alpha_{0+}, \alpha > 0$, on the half-axis R^1_+ in the setting (30) under the following choice of the weight-functions:*

$$\varphi_1(x) = x^{\frac{1}{2}-\lambda}(1+x)^{2\alpha} \ , \quad \psi_1(x) = x^{\frac{1}{2}-\lambda-\alpha}, \tag{32}$$

where $\lambda > \alpha - 1/2, \lambda \neq 0$. The singular system is given by the same σ_m as in (12) and

$$u_m(x) = 2^\lambda a_m x^{\lambda-1/2}(1+x)^{-\lambda-\alpha-1/2} C_m^\lambda\left(\frac{1-x}{1+x}\right) \ , \tag{33}$$

$$v_m(x) = 2^\lambda b_m x^{\lambda+\alpha-1/2}(1+x)^{-\lambda-\alpha-3/2} P_m^{(\lambda-\alpha-1/2,\lambda+\alpha-1/2)}\left(\frac{1-x}{1+x}\right) \tag{34}$$

Theorem 2 . *The asymptotics* $\sigma_m \sim m^{-\alpha/2}$ *takes place for the operator* $I^{\alpha}_{-1}, \alpha > 0$, *on the interval* $[-1,1]$ *in the setting (29) under the following choice of the weight-functions:*

$$\rho_2(x) = (1+x)^{-\beta}(1-x)^{2\alpha+\beta}e^{-\frac{1+x}{1-x}}, \quad r_2(x) = (1+x)^{-\beta-\alpha}(1-x)^{\beta-\alpha}e^{-\frac{1+x}{1-x}},$$

where $\beta > -1$. *The singular system is given by the same* σ_m *as in (15) and*

$$u_m(x) = 2^{1/2}A_m(1+x)^{\beta}(1-x)^{-\alpha-\beta-1}L_m^{(\beta)}\left(\frac{1+x}{1-x}\right) \ ,$$

$$v_m(x) = 2^{1/2}B_m(1+x)^{\alpha+\beta}(1-x)^{-\beta-1}L_m^{(\alpha+\beta)}\left(\frac{1+x}{1-x}\right)$$

Theorems 1 and 2 lead to the following Question:

Question. *Given* $s \in (0,\alpha]$, *does there exist a pair of weight functions* $\rho(x)$ *and* $r(x)$, *such that s-numbers of the operator*

$$I^{\alpha}_0 \ : \ L^2(\Omega;\rho(x)) \to L^2(\Omega; r(x)) \ ,$$

with $\Omega = [0,1]$ *or* $\Omega = R^1_+$ *have the asymptotic*

$$\sigma_m \sim m^{-s} \quad ?$$

Because of Theorems 1 and 2, Question seems to be especially natural for values $s \in [\alpha/2, \alpha]$.

In Theorem 3 below we give some version of Theorem 1 for the multidimensional Riesz potential operator (7) using the notation $\tilde{x} = (x, x_{n+1}) \in R^{n+1}$, where $x \in R^n$, and $s(\tilde{x})$ for the stereographic projection in the Euclidean space R^{n+1} , which maps the hyperplane $R^n = \{\tilde{x} \in R^{n+1} : x_{n+1} = 0\}$, onto the unit sphere $S^n \subset R^{n+1}$, see (21). By $Y_{m\mu}(\xi), \xi \in S^n$, we denote basic spherical harmonics (see Section 1, no 6).).

Theorem 3. The operator

$$I^{\alpha} \ : \ L^2\left(R^n;(1+|x|^2)^{\alpha}\right) \to L^2\left(R^n;(1+|x|^2)^{-\alpha}\right), \ \alpha > 0, \alpha \neq n, n+2, n+4, ..., \tag{35}$$

has singular values

$$\sigma_m = \frac{1}{2^{\alpha}}\frac{\Gamma(m+\frac{n-\alpha}{2})}{\Gamma(m+\frac{n+\alpha}{2})} \ \sim \ m^{-\alpha} \tag{36}$$

and a singular system is $\{\sigma_m, u_{m\mu}(x), v_{m\mu}(x)\}$ with

$$u_{m\mu}(x) = \frac{2^{\frac{n}{2}}\tilde{Y}_{m\mu}(x)}{(1+|x|^2)^{\frac{n+\alpha}{2}}}, \qquad v_{m\mu}(x) = \frac{2^{\frac{n}{2}}\tilde{Y}_{m\mu}(x)}{(1+|x|^2)^{\frac{n-\alpha}{2}}}, \tag{37}$$

$$\mu = 1, 2, ..., d_{n+1}(m); m = 0, 1, 2, ...$$

where $\tilde{Y}_{m\mu}(x) = Y_{m\mu}[s(\tilde{x})]\mid_{x_{n+1}=0}$.

Corollary 1 (of Theorem 3) . The one-dimensional Riesz potential (5) in the setting

$$I^\alpha : L^2(R^1; (1+x^2)^\alpha) \to L^2(R^1; (1+x^2)^{-\alpha}, 0 < \alpha < 1, \qquad (38))$$

has a singular system $\{\sigma_m, u_{m\mu}, v_{m\mu}\}_{\mu=1,2;m\in\mathbf{N}_0}$ where σ_m is given by (36) with $n = 1$ and

$$u_{m1}(x) = \frac{T_m\left(\frac{2x}{1+x^2}\right)}{(1+x^2)^{\frac{1+\alpha}{2}}}, \quad v_{m1}(x) = \frac{T_m\left(\frac{2x}{1+x^2}\right)}{(1+x^2)^{\frac{1-\alpha}{2}}},$$

where $m = 0, 1, 2, ...$ and

$$u_{m2}(x) = \frac{(1-x^2)U_{m-1}\left(\frac{2x}{1+x^2}\right)}{(1+x^2)^{\frac{3+\alpha}{2}}}, \quad v_{m2}(x) = \frac{(1-x^2)U_{m-1}\left(\frac{2x}{1+x^2}\right)}{(1+x^2)^{\frac{3-\alpha}{2}}}$$

where $m = 1, 2, ...$, $T_m(x)$ and $U^m(x)$ being the Chebyshev polynomials of the 1st and 2nd kind, respectively.

Remark 3. According to the above corollary the singular system is found explicitly in the one -dimensional case in terms of Chebyshev polynomials. If $n > 1$, the functions $Y_{m\mu}(x), x \in S^n$, we use in (37) can be only described as restrictions of basic harmonic homogeneous polynomials $Y_{m\mu}(\tilde{x}), \tilde{x} \in R^{n+1}$ onto S^n. In this connection we refer to the paper [17] where there is given a criterion, in terms of coefficients, for an arbitrary homogeneous polynomial $\sum_{|j|=m} a_j x^j$ to be harmonic.

Corollary 2 (of Theorem 3) . Let $0 < \alpha < 1/2$. The singular values of the Liouville fractional integration operators (3)-(4) considered in the setting (38) , have the asymptotics $\sigma_m \sim m^{-\alpha}$.

Theorem 4. *The potential-type operator (6) in the setting*

$$A^\alpha \ : \ L^2\left([-1,1]; (1-x^2)^\alpha\right) \to L^2\left([-1,1]; (1-x^2)^{-\alpha}\right), \tag{39}$$

$0 < \alpha < 1$, *has the singular system* $\{\sigma_m, u_m, v_m\}_{m\in\mathbf{N_0}}$ *with*

$$\sigma_m = \frac{\pi\Gamma(m+1-\alpha)}{m!\sin\frac{\alpha\pi}{2}\Gamma(1-\alpha)} \sim \frac{1}{m^\alpha}$$

and

$$u_m(x) = c_m(1-x^2)^{-\alpha}C_n^{\frac{1-\alpha}{2}}(x)\ , \quad v_m(x) = c_m C_n^{\frac{1-\alpha}{2}}(x) \tag{40}$$

with $c_m = \Gamma\left(\frac{1-\alpha}{2}\right)2^{-\frac{\alpha}{2}}\left(\frac{m!(n+\frac{1-\alpha}{2})}{\pi\Gamma(n+1-\alpha)}\right)^{\frac{1}{2}}$, $C_m^\lambda(x), \lambda = \frac{1-\alpha}{2}$, *being the Gegenbauer polynomials.*

4. Proofs of Theorems.

Proof of Theorem 1.

Naturally, we base our proof on reduction to Theorem A′ . Therefore, we use the change of variables (8) and the mapping operator (9) to reduce the case of the half-axis R^1_+ to the case of the interval [-1,1] and then apply Theorem A′ . By (1) we have

$$\int_0^x \frac{u(t)dt}{(x-t)^{1-\alpha}} = 2^\alpha(1+y)^{1-\alpha}\int_y^1 \frac{u\left(\frac{1-\tau}{1+\tau}\right)}{(1+\tau)^{1+\alpha}}\,\frac{d\tau}{(\tau-y)^{1-\alpha}} \tag{41}$$

with $y = \frac{1-x}{1+x}$. This relation is crucial for our purposes (compare with the relation (5.33) in [19]).

Because of Theorem A′ we choose the space

$$L^2\left([-1,1];\left(\frac{1-x}{1+x}\right)^{1/2-\lambda}\right)$$

for functions on [-1,1] . So, we base our consideration on Lemma 5 with $\mu = -\nu = \lambda - 1/2$. For the mapping (9), by Lemma 5 we see that

$$u^*(\tau) = 2^{1/2}(1+\tau)^{-1-\alpha}u\left(\frac{1-\tau}{1+\tau}\right) \in L^2\left([-1,1];\left(\frac{1-\tau}{1+\tau}\right)^{1/2-\lambda}\right) \tag{42}$$

if $u(x) \in L^2\left(R^1_+; x^{-\lambda+1/2}(1+x)^{2\alpha}\right)$. In terms of the mapping (9) the relation (41) can be represented as

$$I^\alpha_{0+}u(x) = 2^{\alpha-1/2}(1+y)^{1-\alpha}I^\alpha_{1-}u^*(y), \tag{43}$$

which can be also given symmetrically as

$$I_{1-}^{\alpha} v(y) \;=\; 2^{\alpha-1/2}(1+x)^{1-\alpha} I_{0+}^{\alpha} v^*(x). \tag{44}$$

The latter can be verified directly in view of (10).

To apply Theorem A′ , we must be sure that our consideration of the operator I_{0+}^{α} as

$$I_{0+}^{\alpha} \;:\; L^2\left(R_+^1; x^{-\lambda+1/2}(1+x)^{2\alpha}\right) \to L^2\left(R_+^1; x^{-\lambda-\alpha+1/2}\right) \tag{45}$$

exactly corresponds to the setting (17) of the operator I_{1-} in Theorem A′. (We note that the consideration (17) under the condition $\lambda > \alpha - 1/2$ is in accordance with Lemma 2 . (Lemma 2 gives even a stronger assertion).

Checking this correspondence , in view of (43) we have:

$$\|I_{0+}^{\alpha} u(x)\|_{L_2(R_+^1; x^{-\lambda-\alpha+1/2})} \;=\; 2^{\alpha-1/2}\left\{\int_0^{\infty} \left|(1+y)^{1-\alpha} I_{1-}^{\alpha} u^*(y)\right|^2_{y=\frac{1-x}{1+x}} x^{-\lambda-\alpha+1/2} dx\right\}^{1/2}$$

$$= 2^{\alpha}\left\{\int_{-1}^{1} \left|(1+y)^{-\alpha} I_{1-}^{\alpha} u^*(y)\right|^2 \left(\frac{1-y}{1+y}\right)^{-\lambda-\alpha+1/2} dy\right\}^{1/2}$$

$$= 2^{\alpha}\|I_{1-}^{\alpha} u^*(y)\|_{L^2\left([-1,1]; \left(\frac{1-y}{1+y}\right)^{-\lambda+1/2}(1-y^2)^{-\alpha}\right)}.$$

Then, by Theorem A′ we have

$$I_{1-} u_m(y) \;=\; \sigma_m v_m(y)$$

where σ_m is given by (18) and $u_m(y), v_m(y)$ are defined in (19). Applying the connection (43), we obtain

$$2^{\alpha-1/2}(1+x)^{1-\alpha} I_{0+}^{\alpha} u_m(x) = \sigma_m v_m\left(\frac{1-x}{1+x}\right)$$

or

$$I_{0+}^{\alpha} u_m^*(x) = \sigma_m v_m^{\#}(x) \tag{46}$$

where

$$v_m^{\#}(x) = \frac{(1+x)^{2\alpha}}{2^{\alpha}} v_m^*(x).$$

Here the sequence $\{u_m(x)\}$ is an orthonormal basis in the space $L^2\left(R^1_+; x^{-\lambda+1/2}(1+x)^{2\alpha}\right)$ by Lemma 5. The orthogonality also holds for $v^{\#}_m(x)$ in $L^2\left(R^1_+; x^{-\lambda-\alpha+1/2}\right)$. Really,

$$\int_0^\infty v^{\#}_k(x)v^{\#}_m(x)x^{-\lambda-\alpha+1/2}dx =$$

$$\int_{-1}^1 v_k(y)v_m(y)\left(\frac{1-y}{1+y}\right)^{-\lambda+1/2}(1-y^2)^{-\alpha}dy = \delta_{mk}.$$

Then (46) means that

$$\{\sigma_m, u^*_m(x), v^{\#}_m(x)\}$$

is a singular system for the operator (45). The direct calculation shows that $u^*_m(x)$ and $v^{\#}_m(x)$ coincide with functions given in (33)-(34). $\square$

P r o o f   o f   T h e o r e m   2 .

The proof of Theorem 2 is completely analogous to that of Theorem 1 : we must

1) use the relation (41) = (43) to pass from the interval to the half-axis;

2) to apply then Theorem A for the half-axis and return back to the interval by means of (8) , recalculating $u_m(x)$ and $v_m(x)$;

3) to verify that the new constructed sequences $u_m(x)$ and $v_m(x)$ are really orthonormal in the corresponding spaces.

P r o o f   o f   T h e o r e m   3 .

We base the proof on the following two facts:

a) there exists an explicit relation between the Riesz potential (7) over R^n and the similar spherical potential over $S^n \subset R^{n+1}$;

b) the $L^2(S^n)$-basis of spherical harmonics is the set of eigen-functions of the spherical potential operator.

The relation mentioned in a) is the following

$$\int_{S^n}\frac{f(\sigma)d\sigma}{|\xi-\sigma|^{n-\alpha}} = 2^\alpha(|x|^2+1)^{\frac{n-\alpha}{2}}\int_{R^n}\frac{f[s(y)]dy}{|x-y|^{n-\alpha}(1+|y|^2)^{\frac{n+\alpha}{2}}}, \tag{47}$$

where $\sigma = s(y)$ is the stereographic projection (21) of R^n onto the unit sphere S^n. The equality (47) immediately follows from (22)-(23) (see [15] and references there).

As regards the fact b), it is given precisely by the relation

$$\frac{1}{\gamma_n(\alpha)} \int_{S^n} \frac{Y_m(\sigma)d\sigma}{|\xi - \sigma|^{n-\alpha}} \; = \; \lambda_m \; Y_m(\xi) \; , \; \xi \in S^n, \tag{48}$$

where $Y_m(\xi)$ is an arbitrary spherical harmonic of order m and

$$\lambda_m = \frac{\Gamma\left(m + \frac{n-\alpha}{2}\right)}{\Gamma\left(m + \frac{n+\alpha}{2}\right)}. \tag{49}$$

The formula (48)-(49) is known: it can be found e.g. in [18] in a non-direct form. We mention that this formula is a consequence of the Funk-Hekke formula (25). Really, since

$$|\xi - \sigma|^2 = 2(1 - \xi\sigma)$$

for $\xi, \sigma \in S^n$, the spherical potential operator, that is the left-hand side in (47) , has the form (26) with

$$k(t) = \frac{(1-t)^{\frac{\alpha-n}{2}}}{2^{\frac{n-\alpha}{2}}\gamma_n(\alpha)}.$$

Then the Funk-Hekke formula (25) is applicable, which gives (48) and, to obtain (49), it remains to calculate the integral (27) for the above kernel $k(t)$:

$$\lambda_m \; = \; \frac{2^{1+\frac{\alpha-n}{2}}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)\gamma_n(\alpha)\begin{pmatrix} m+n-2 \\ m \end{pmatrix}} \int_{-1}^{1} (1-t)^{\frac{\alpha-2}{2}}(1+t)^{\lambda-\frac{1}{2}} C_m^\lambda(t)dt \; , \; \lambda = \frac{n-1}{2},$$

which is known:

$$\int_{-1}^{1} (1+t)^\beta (1-t)^{\lambda-\frac{1}{2}} C_m^\lambda(t)dt \; =$$

$$\frac{2^{\beta+\lambda+1/2}\Gamma)\beta+1)\gamma(\lambda+1/2)\Gamma(2\lambda+m)\Gamma(\beta-\lambda+3/2)}{m!\Gamma(2\lambda)\Gamma(\beta-\lambda-m+3/2)\Gamma(\beta+\lambda+m+3/2)},$$

see e.g. [8], 7.311.3. Hence, after easy calculations we arrive at (49).

Now, we use the relation (47) which allows to transform (48) to

$$\frac{2^\alpha}{\gamma_n(\alpha)} \int_{R^n} \frac{Y_m[s(y)]}{(1+|y|^2)^{\frac{n+\alpha}{2}}} \frac{dy}{|x-y|^{n-\alpha}} \; = \; \lambda_m \frac{Y_m[s(x)]}{(1+|x|^2)^{\frac{n-\alpha}{2}}}.$$

This is valid for an arbitrary spherical harmonic Y_m. We take in particular basic harmonics and then this may be rewritten as

$$I^\alpha u_{m\mu}(x) \;=\; \frac{\lambda_m}{2^\alpha} v_{m\mu}(x)$$

with the functions from (37). These functions form basises in the spaces

$$L^2(R^n;(1+|x|^2)^\alpha) \quad , \quad L^2(R^n;(1+|x|^2)^{-\alpha}) \;, \tag{50}$$

respectively. Checking that, for example, for the former of these spaces, by (23) we have

$$\int_{R^n} u_{m\mu}(x) u_{k\nu}(x)(1+|x|^2)^\alpha dx =$$

$$2^n \int_{R^n} \frac{Y_{m\mu}[s(x)] Y_{k\nu}[s(x)]}{(1+|x|^2)^n} dx \;=\; \int_{S^n} Y_{m\mu}(\xi) Y_{k\nu}(\xi) d\xi = \delta_{m\mu,k\nu}$$

where $\delta_{m\mu,k\nu} = 1$ if both $m = k$ and $\mu = \nu$ and $\delta_{m\mu,k\nu} = 0$ otherwise . This gives the orthogonality condition. It remains to refer to the fact that $\{Y_{m\mu}(\sigma)\}_{\mu=1,\ldots,d_{n+1}(m);m=0,1,2,\ldots}$ is a basis in $L^2(S^n)$ and so the functions (37) form basis in the corresponding weighted spaces (50) as well.

Finally, it remains to observe that the operator I^α is bounded in the setting (35) which follows from the relation (47) and boundedness of the operator in the left hand side of (47) in the space $L^2(S^n)$. $\square$

Proof of Corollary 1

The operator I^α is bounded in the setting (38) by Lemma 4. So, we should only calculate the functions $Y_{m\mu}(x)$ in (37) , taking into account that $\mu = 1, 2$ in case $n = 1$. Evidently,

$$Y_{m1}(\tilde{x}) \mid_{S^1} = \frac{\cos m\theta}{\sqrt{2}} \;,\quad Y_{m2}(\tilde{x}) \mid_{S^1} = \frac{\sin m\theta}{\sqrt{2}} \quad, \; 0 < \theta < 2\pi$$

for $n+1 = 2(\tilde{x} = (x, x_2) \in S^1 = \{\tilde{x} \in R^2 : x^2 + x_2^2 = 1\}; \theta = \arccos x)$. Since

$$s(\tilde{x}) = \left(\frac{2x}{1+x^2+x_2^2}, \frac{x^2+x_2^2-1}{x^2+x_2^2+1} \right),$$

we obtain

$$Y_{m1}[s(\tilde{x})] \mid_{x_2=0} = \frac{1}{\sqrt{2}} T_m \left(\frac{2x}{1+x^2} \right) \;,$$

$$Y_{m2}[s(\tilde{x})] \mid_{x_2=0}= \frac{1}{\sqrt{2}}\frac{1-x^2}{1+x^2}U_{m-1}\left(\frac{2x}{1+x^2}\right) \ ,$$

which proves the corollary.

P r o o f o f C o r o l l a r y 2 .

It is known ([19], Section 12) that the Riesz potential operator (5) and the Liouville fractional integration operators (3)-(4) are connected with each other by means of the relation

$$I^{\alpha}_{\pm} \ = \ I^{\alpha}B_{\pm} \ = \ B_{\pm}I^{\alpha} \tag{51}$$

where

$$B_{\pm} = \cos\frac{\alpha\pi}{2} \ \ E \ \mp \ \sin\frac{\alpha\pi}{2} \ S \ ,$$

E being the identity operator and

$$Su \ = \ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)dt}{t-x}.$$

It is well known that $S^2=-E$, see [13]. Then, evidently, the operators $B_{\pm}$ are invertible and $(B_{\pm})^{-1}=B_{\mp}$, so that

$$I^{\alpha} = B_{m}I^{\alpha}_{\pm} \tag{52}$$

We apply Proposition 1 (Section 1, no 4) to (51) and (52) and obtain

$$\sigma_m(I^{\alpha}_{\pm}) \leq c(\alpha)\sigma_m(I^{\alpha}) \ ,$$

$$\sigma_m(I^{\alpha}) \leq c(\alpha)\sigma_m(I^{\alpha}_{\pm}) \ ,$$

where $c(\alpha) = |\cos\frac{\alpha\pi}{2}| + |\sin\frac{\alpha\pi}{2}| \|S\|_{L^2(R^1;(1+x^2)^{\alpha})}$.

It remains to refer to the boundedness of the operator S in the weighted spaces $L^2(R^1;(1+x^2)^{\pm\alpha})$ in case $0<\alpha<1/2$.

P r o o f o f T h e o r e m 4 .

The following relation holds

$$\int_{-1}^{1}|x-y|^{\alpha-1}(1-y^2)^{-\frac{\alpha}{2}}C_m^{\frac{1-\alpha}{2}}(y)dy \ = \ \sigma_m C_m^{\frac{1-\alpha}{2}}(x), \ |x|\leq 1,$$

where $0 < \alpha < 2, \sigma_m = \frac{2\Gamma(\alpha)\Gamma(m+1-\alpha)}{m! \cos\frac{\alpha\pi}{2}}$, see [16] (where it is given in a non-direct form). This may be rewritten as

$$A^\alpha u_m \ = \ \sigma_m \ v_m,$$

where u_m and v_m are the functions given in (40). Both $\{u_m\}$ and $\{v_m\}$ form a basis in the corresponding space. This is checked directly due to the well known orthogonality relations for the Gegenbauer polynomials. It remains to refer to the boundedness of the operator $A^\alpha = \Gamma(\alpha)(I^\alpha_{-1+} + I^\alpha_{1-})$ in the setting (39) which follows from Lemmas 1-2 in the case $0 < \alpha < 1$.

5. Returning to the Question.

Let us reformulate the Question put in Section 3 in the following form.

Given $s \in (0, \alpha]$ *, does there exist a pair of Hilbert spaces* U *and* V *of functions defined on* $[0, 1]$ *or* R^1_+ *such that singular values of the operator* $I^\alpha_{0+} : U \to V$ *have the asymptotics*

$$\sigma_m \ \sim \ m^{-s} \ ?$$

Naturally, in such a general setting this question is already very easy, definitely having a positive answer. Really, admitting spaces U and V of types completely independent of each other, we can always take the space V with better smoothness properties, which will immediately diminish order of decreasing of singular numbers. This is quite understandable in general, but to be precise, we give below the exact statement (Theorem 5).

Let $H^s = H^s(R^1) = H^{s,2}(R^1)$ be the well known Sobolev space of fractional smoothness, known also as the space of Bessel potentials. It is defined as the completion of C^∞_0 with respect to the norm

$$\|u\|_{H^s} \ = \ \|u\|_2 \ + \ \|F^{-1}(1 + |\xi|^2)^{1/2} F u\|_2 \tag{53}$$

where F stands for the Fourier transform operator.

By $H^s_0([0, 1])$ we denote the space of restrictions of functions $u(x) \in H^s(R^1)$ onto [0,1] satisfying the conditions

$$u^{(k)}(0) = 0, \quad k = 0, 1, ..., [s - 1/2].$$

$(H^0_0 = L^2)$.

Theorem 5. *Given any $s \in [0,\alpha], s \neq \alpha - 1/2, \alpha - 3/2, ...,$ the singular numbers of the operator*

$$I^{\alpha}_{0+} \;:\; L^2([0,1]) \to H^{\alpha-s}_0([0,1]) \tag{54}$$

have the asymptotics $\sigma_m \;\sim\; m^{-s}$.

Proof. The key moment in the proof is in the following fact: if $s \neq 1/2, 3/2, ..., [\alpha - 1/2]$, the space $H^s_0([0,1])$ coincides with the range $I^s_{0+}(L^2)$ of the fractional integration operator I^s_{0+} , considered in $L^2([0,1])$, up to equivalence of the norm (53) to the natural norm in the range defined by

$$\|u\|_{I^s_{0+}(L^2)} \;=\; \|D^s_{0+}u\|_{L^2},$$

where D^{α}_{0+} is the fractional differentiation operator, left inverse to I^{α}_{0+} (see [19], Theorem 18.3 and Remark 18.1).

Therefore, the range $H^{\alpha-s}_0([0,1])$ used in (54) can be treated as a Hilbert space with the scalar product

$$(u,v)_V = (D^{\alpha-s}_{0+}u, D^{\alpha-s}_{0+}v)_{L^2}. \tag{55}$$

The operator $I^s_{0+} : L^2([0,1]) \to L^2([0,1])$ is known to have the singular system $\{\sigma_m, u_m, v_m\}$ with

$$\sigma_m \;\sim\; m^{-s}$$

and some $\{u_m(x)\}$ and $\{v_m(x)\}$:

$$I^s_{0+}u_m(x) = \sigma_m v_m(x).$$

Then we take $U_m(x) = u_m(x)$, $V_m(x) = I^{\alpha-s}_{0+}v_m(x)$. Obviously,

$$I^{\alpha}_{0+}U_m(x) = \sigma_m V_m(x)$$

and it remains to note that the sequence $V_m(x)$ is orthonormal in $H^{\alpha-s}_0$ with respect to the scalar product (55). □

6. Further comments.

In view of Theorems 1 and 2 it is interesting to find whether there exist weight function $\rho_\lambda(x)$ and $r_\lambda(x)$ depending, say, on a parameter λ such that the operator

$$I^{\alpha}_{0+} \;:\; L^2(\Omega; \rho_\lambda(x)) \to L^2(\Omega; r_\lambda(x)) \tag{56}$$

has singular values with behaviour at infinity explicitly depending on the parameter λ.

This remains open for the fractional integration operator . Lemma 6 below shows that such an effect happens for an integral operator of another kind, with essentially better behaviour of the kernel. Anyhow, we find it important to show that such things really may happen in principle.

Let us consider the example

$$Ku(x) \;=\; \int_{-\infty}^{\infty} e^{-(x-y)^2} u(y)dy, \;\; x \in R^1. \tag{57}$$

Since the kernel e^{-x^2} is infinitely differentiable, its s-numbers vanish at infinity more rapidly than any power. It is natural to expect that

$$\sigma_m \;\sim\; q^m \tag{58}$$

for some $q \in (0,1)$. The following lemma shows that the number q determining the asymptotics (58), explicitly depends on the choice of weight functions.

We treat the operator (58) as

$$K \;:\; L^2(R^1; e^{-\beta^2 x^2}) \to L^2(R^1; e^{-\gamma^2 x^2}) \tag{59}$$

where β is an arbitrary number in (0,1) and $\gamma = \frac{\beta}{(1-\beta^2)^{1/2}}$.

Lemma 6. *The singular values of the operator (59) are equal to*

$$\sigma_n \;=\; \pi^{1/2} q^{n+1/2} \tag{60}$$

where $q = (1-\beta^2)^{1/2}$ *and*

$$u_m(x) \;=\; \left(\frac{\beta}{2^m m! \pi^{1/2}}\right)^{1/2} H_m(\beta x), \tag{61}$$

$$v_m(x) \;=\; \left(\frac{\gamma}{2^m m! \pi^{1/2}}\right)^{1/2} H_m(\gamma x), \tag{62}$$

the $H_n(x)$ *being the Hermite polynomials.*

Proof. The Hermite polynomials form an orthogonal basis in the space $L^2(R^1; e^{-x^2})$:

$$\int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_m(x) dx = c_m \delta_{mk} \;, \quad c_m = 2^m m! \pi^{1/2}.$$

Hence it is easily derived that the sequences (61) - (62) are orthonormal basises in the spaces $L^2(R^1; e^{-\beta^2 x^2})$ and $L^2(R^1; e^{-\gamma^2 x^2})$, respectively. So, it remains to check the relation $Ku_m = \sigma_m v_m$ that is

$$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(\beta y)dy \;=\; \pi^{1/2}(1-\beta^2)^{m/2} H_m(\gamma x) \tag{63}$$

which is known, see [8], 7.374.8, but may be also checked directly. Really, we may make use of the generating function for the Hermite polynomials:

$$e^{-t^2+2ty} \;=\; \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(y) \tag{64}$$

(see [8], 8.957.1). Replacing here y by αy , multiplying by $e^{-(x-y)^2}$ and integrating and then using the generating series (64) again, we arrive at (63). □

Remark 4. It is well known that convolution operators with a summable kernel definitely are not compact in $L^p(R^1), 1 \le p \le \infty$, see e.g. [9]. Since $lim_{n\to\infty}\sigma_n = 0$ if and only if an operator is compact ([4], p.62, Corollary 7.1), this means that the operator K in the setting (59) should be certainly compact. The latter may be checked directly. Really, by Remark 2, the setting (59) is equivalent to $K_1: \; L^2(R^1) \to L^2(R^1)$ for

$$K_1 u = e^{-\frac{\gamma^2 x^2}{2}} \int_{-\infty}^{\infty} e^{\frac{\beta^2 t^2}{2}} e^{-(x-t)^2} u(t)dt \; .$$

We transform K_1 to

$$K_1 u = e^{-c^2 x^2} \int_{-\infty}^{\infty} e^{-(bx-ay)^2} u(t)dt$$

with $a = (1-\frac{\beta^2}{2})^{1/2}, \; b = 1/a, \; c = \beta^2 \left[2(1-\beta^2)(2-\beta^2)\right]^{-1/2}$. Then the operator K_1 is compact because of the factor $e^{-c^2x^2}$ (see e.g. [9]).

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