# THE TOPOLOGICAL UNIVERSE OF LOCALLY PRECOMPACT SEMIUNIFORM CONVERGENCE SPACES

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In memory of Professor K. Morita

**Abstract.** The importance of local precompactness has become apparent in theorems of the Ascoli type proved by Bentley and Herrlich [3] in the realm of filter spaces as well as by Wyler [14] in the realm of uniform limit spaces. Here local precompactness is studied in the highly convenient setting of semiuniform convergence spaces which form a common generalization of (symmetric) limit spaces (and thus of symmetric topological spaces ), filter spaces and uniform limit spaces (and thus of uniform spaces). It turns out that it leads to a topological universe. Furthermore, locally precompact semiuniform convergence spaces are exactly the precompactly generated semiuniform convergence spaces, and the relationships to locally compact, compact and precompact semiuniform convergence spaces respectively, studied before in [9] and [10], are clarified.

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| Semiuniform convergence spaces | filter spaces |
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## 1. Introduction

Recently semiuniform convergence spaces have been studied by the author (cf. [8] and [9]) as a common generalization of (symmetric) limit spaces (and thus of symmetric topological spaces) as well as of uniform limit spaces (and thus of uniform spaces) with many convenient properties such as cartesian closedness, hereditariness and the fact that products of quotients are quotients (cf. also [2]). They form the suitable framework for studying continuity, Cauchy continuity and uniform continuity as well as convergence structures in function spaces, namely simple convergence, continuous convergence and uniform convergence.

In this article the localization of precompactness (= total boundedness) is introduced in the realm of semiuniform convergence spaces. It turns out that the category **LPC-SUConv** of locally precompact semiuniform convergence spaces (and uniformly continuous maps) is a topological universe (i.e. a topological category which is cartesian closed and hereditary) containing all precompact semiuniform convergence spaces. In particular,

in a topological universe quotients are finitely productive and hereditary. Furthermore, **LPC-SUConv** is the bicoreflective hull of all precompact semiuniform convergence spaces (formed in **SUConv**), in other words: a semiuniform convergence space is locally precompact iff it is precompactly generated.
The localization of precompactness is more a topological procedure than a uniform one, namely in uniform spaces there is no difference between precompactness and local precompactness (cf. 3.6.), whereas in regular topological spaces local precompactness means local compactness (cf. 3.4.) which differs from compactness (= precompactness according to [9;4.4.]).
The significance of local precompactness has been demonstrated by H.L. Bentley and H. Herrlich [3] for filter spaces and by O. Wyler [14] for uniform limit spaces, namely the mentioned authors proved theorems of the Ascoli type in which local precompactness plays an essential role. The terminology of this article corresponds to [1] and [7].

**Conventions.** 1) A filter on a set $X$ is not allowed to contain the empty set $\emptyset$.
2) Subcategories are always assumed to be full and isomorphism–closed.

## 2. Some basic definitions and results

**2.1 Definitions.** 1) A *semiuniform convergence space* is a pair $(X, \mathcal{J}_X)$ where $X$ is a set and $\mathcal{J}_X$ a set of filters on $X \times X$ such that the following are satisfied:

$UC_1)$ $\dot{x} \times \dot{x} \in \mathcal{J}_X$ for each $x \in X$, where $\dot{x} = \{A \subset X : x \in X\}$,

$UC_2)$ $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{F} \subset \mathcal{G}$,

$UC_3)$ $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\} \in \mathcal{J}_X$, where $F^{-1} = \{y, x) : (x, y) \in F\}$.

2) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *diagonal* provided that the filter $(\Delta_X)$ generated by the diagonal $\Delta_X$ of $X \times X$ belongs to $\mathcal{J}_X$.

3) A map $f : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ between semiuniform convergence spaces is called *uniformly continuous* provided that $f \times f(\mathcal{J}_X) \subset \mathcal{J}_Y$.

4) The category of semiuniform convergence spaces (and uniformly continuous maps) is denoted by **SUConv**.

5) The object class of the full subcategory **ULim** of **SUConv** consists of all *uniform limit spaces*, i.e. of all semiuniform convergence spaces $(X, \mathcal{J}_X)$ satisfying the following two additional conditions:

$UC_4)$ $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}_X$,

$UC_5)$ $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \circ \mathcal{G} \in \mathcal{J}_X$ (whenever $\mathcal{F} \circ \mathcal{G}$ exists, i.e. $F \circ G = \{(x, y) : \exists z \in X \text{ with } (x, z) \in G \text{ and } (z, y) \in F\} \neq \emptyset$ for every $F \in \mathcal{F}, G \in \mathcal{G}$), where $\mathcal{F} \circ \mathcal{G}$ denotes the filter generated by the filter base $\{F \circ G : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$.

**2.2 Remarks.** 1) The category **Unif** of uniform spaces (and uniformly continuous maps) in the sense of A. Weil [13] can be nicely embedded into the category **SUConv**, namely

**Unif** is isomorphic to the full subcategory **PrULim** of **ULim** whose object class consists of all *principal uniform limit spaces* (a uniform limit space $(X, \mathcal{J}_X)$ is called a principal uniform limit space provided that there is a filter $\mathcal{F}$ on $X \times X$ such that $\mathcal{J}_X = [\mathcal{F}]$, where $[\mathcal{F}] = \{\mathcal{G} : \mathcal{G}$ is a filter on $X \times X$ with $\mathcal{G} \supset \mathcal{F}\}$). Since **PrULim** is a bireflective subcategory of **ULim** (cf. [7;4.2.8.1]) and **ULim** is a bireflective subcategory of **SUConv** (cf. [8;2.2]) **Unif** can be embedded into **SUConv** as a bireflective subcategory.

2) **SUConv** is a topological category, in which initial and final structures can be easily described (cf. e.g. [9;2.2.2)]).

**2.3 Definitions.** 1) A *Kent convergence space* is a pair $(X, q)$, where $X$ is a set and $q \subset F(X) \times X$, where $F(X)$ denotes the set of all filters on $X$, such that the following are satisfied:

$C_1)$ $(\dot{x}, x) \in q$ for each $x \in X$,

$C_2)$ $(\mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{F} \subset \mathcal{G}$,

$C_3)$ $(\mathcal{F} \cap \dot{x}, x) \in q$, whenever $(\mathcal{F}, x) \in q$.

If $(X, q)$ is a *Kent convergence space*, then we say that $\mathcal{F} \in F(X)$ *converges* to $x \in X$ provided that $(\mathcal{F}, x) \in q$.

2) A Kent convergence space $(X, q)$ is called *symmetric* provided that the following is satisfied:

$$(\mathcal{F}, x) \in q \quad \text{and} \quad y \in \bigcap_{F \in \mathcal{F}} F \quad \text{imply} \quad (\mathcal{F}, y) \in q\ .$$

3) A map $f : (X, q) \to (X', q')$ between Kent convergence spaces is called *continuous* provided that $(f(\mathcal{F}), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$.

**2.4 Remarks.** 1)a) Every semiuniform convergence spaces has an *underlying (symmetric) Kent convergence space* $(X, q_{\gamma_{\mathcal{J}_X}})$ defined as follows:

$$(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}} \Longleftrightarrow \mathcal{F} \cap \dot{x} \in \gamma_{\mathcal{J}_X}\ ,$$

where $\gamma_{\mathcal{J}_X} = \{\mathcal{H} \in F(X) : \mathcal{H} \times \mathcal{H} \in \mathcal{J}_X\}$.

b) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a *convergence space* provided that

$$\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : \text{ there are some } x \in X \text{ and } \mathcal{G} \in F(X) \text{ with } (\mathcal{G}, x) \in q_{\gamma_{\mathcal{J}_X}} \text{ such that } \mathcal{F} \supset \mathcal{G} \times \mathcal{G}\},$$

in other words: A convergence space is a semiuniform convergence space which is generated by all convergent filters. The category **Conv** of convergence spaces (and uniformly continuous maps) is isomorphic to the category $\mathbf{KConv}_s$ of symmetric Kent convergence spaces (and continuous maps); it is a bicoreflective subcategory of **SUConv** (cf. [9;3.2.,3.3. and 3.4.]) and thus closed under formation of quotients and coproducts (formed in **SUConv**).

2) If $(X, q)$ is a Kent convergence space, then for each subset $A$ of $X$ we define the *closure* $\overline{A}$ of $A$ by

$$\overline{A} = \{x \in X : \text{ there is some } \mathcal{G} \in F(X) \text{ with } (\mathcal{G}, x) \in q \text{ and } A \in \mathcal{G}\}$$

whereas the *topological closure* $\overline{A}^t$ of $A$ is defined by

$$\overline{A}^t = \bigcap \{B \subset X : B = \overline{B} \text{ and } B \supset A\}$$

3) **Conv** is closed under formation of closed subspaces and products (formed in **SUConv**) [9;3.13.].

**2.5 Definition.** A semiuniform convergence space $(X, \mathcal{J}_X)$ is called t–*regular* provided that for each $\mathcal{F} \in \mathcal{J}_X$ the subfilter $\overline{\mathcal{F}}^t$, generated by the filter base of all closed elements of $\mathcal{F}$ (i.e. all $F \in \mathcal{F}$ with $F = \overline{F}$ or equivalently $F = \overline{F}^t$), belongs to $\mathcal{J}_X$.

**2.6 Remark.** Every uniform space (= principal uniform limit space) is $t$–regular.

**2.7 Definitons.** 1) A *filtermerotopic space* (shortly: a *filter space*) is a pair $(X, \gamma)$, where $X$ is a set and $\gamma \subset F(X)$ such that the following are satisfied:

(1) $\dot{x} \in \gamma$ for each $x \in X$,

(2) $\mathcal{G} \in \gamma$ whenever $\mathcal{F} \in \gamma$ and $\mathcal{F} \subset \mathcal{G}$.

2) A map $f : (X, \gamma) \to (X', \gamma')$ between filter spaces is called *Cauchy continuous* provided that the filter $f(\mathcal{F})$ generated by $\{f[F] : F \in \mathcal{F}\}$ belongs to $\gamma'$ for each $\mathcal{F} \in \gamma$.

3) The category of filter spaces (and Cauchy continuous maps) is denoted by **Fil**.

**2.8 Remarks.** 1) Every semiuniform convergence space $(X, \mathcal{J}_X)$ has an *underlying filter space* $(X, \gamma_{\mathcal{J}_X})$ defined as follows

$$\gamma_{\mathcal{J}_X} = \{\mathcal{F} \in F(X) : \mathcal{F} \times \mathcal{F} \in \mathcal{J}_X\} .$$

The elements of $\gamma_{\mathcal{J}_X}$ are called $\mathcal{J}_X$–*Cauchy filters*.

2) If a semiuniform convergence space $(X, \mathcal{J}_X)$ is called **Fil**–*determined* provided that $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : \text{there is some } \mathcal{G} \in \gamma_{\mathcal{J}_X} \text{ with } \mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$ (i.e. a semiuniform convergence space $(X, \mathcal{J}_X)$ is **Fil**–determined iff it is 'generated' by all $\mathcal{J}_X$–Cauchy filters), then the category **Fil–D–SUConv** of all **Fil**–determined semiuniform convergence spaces (and uniformly continuous maps) is isomorphic to **Fil** and a bireflective and bicoreflective subcategory of **SUConv** (cf. [9;2.4.]).

3) The category **Prox** of proximity spaces (and $\delta$–maps) in the sense of Efremovič [5] can be embedded into **Fil** as a bireflective subcategory (cf. [8;2.4.1)] and [7]) whereas the category **Conv** can be embedded into **Fil** as a bicoreflective subcategory (cf. [9;2.8., 2.11.3) and 3.2.]).

**2.9 Definitions.** 1)a) A filter space $(X, \gamma)$ is called *precompact* (or totally bounded) provided that each ultrafilter $\mathcal{U}$ on $X$ belongs to $\gamma$.

b) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *precompact* (or totally bounded) provided that the underlying filter space $(X, \gamma_{\mathcal{J}_X})$ is precompact, i.e. each ultrafilter $\mathcal{U}$ on $X$ is a $\mathcal{J}_X$–Cauchy filter.

2) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *complete* provided that each $\mathcal{J}_X$–Cauchy filter converges in $(X, q_{\gamma_{\mathcal{J}_X}})$.

3)a) A Kent convergence space $(X, q)$ is called *compact* provided that each ultrafilter $\mathcal{U}$ on $X$ converges in $(X, q)$.

b) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *compact* provided that the underlying Kent convergence space $(X, q_{\gamma_{\mathcal{J}_X}})$ is compact.

4) A subset $A$ of a semiuniform convergence space $(X, \mathcal{J}_X)$ is called *compact* (resp. *precompact*) provided that $A$ is compact (resp. precompact) as a subspace.

**2.10 Proposition.** *A filter space $(X, \gamma)$ is precompact iff for each filter $\mathcal{F}$ on $X$ there is some $\mathcal{G} \in \gamma$ with $\mathcal{F} \subset \mathcal{G}$.*

**2.11 Proposition.** *Let $f : (X, \gamma) \to (X', \gamma')$ be a surjective Cauchy continuous map between filter spaces. If $(X, \gamma)$ is precompact, then $(X', \gamma')$ is precompact.*

**Proof.** Let $\mathcal{F} \in F(X')$. Then $f^{-1}(\mathcal{F}) \in F(X)$. By assumption there is some $\mathcal{G} \in \gamma$ with $f^{-1}(\mathcal{F}) \subset \mathcal{G}$. Thus, $\mathcal{F} = f(f^{-1}(\mathcal{F})) \subset f(\mathcal{G})$, where $f(\mathcal{G}) \in \gamma'$.

**2.12 Defintions.** 1) A topological category $\mathcal{C}$, i.e. a concrete category over the category **Set** of sets (and maps) which is initially complete and small–fibred such that all constant maps between its objects are morphisms, is called *hereditary* provided that in $\mathcal{C}$ final (epi–) sinks are hereditary, i.e. if $(f_i : A_i \to A)_{i \in I}$ is a final (epi–) sink in $\mathcal{C}$, $B$ a subspace (=extremal subobject) of $A$, $B_i$ a subspace of $A_i$ with underlying set $f_i^{-1}[B]$ and $g_i : B_i \to B$ is the corresponding restriction of $f_i$, then $(g_i : B_i \to B)_{i \in I}$ is a final (epi–) sink in $\mathcal{C}$ too.

2) A topological category $\mathcal{C}$ is called a *topological universe* provided that one of the following two equivalent conditions is satisfied:

(1) $\mathcal{C}$ is cartesian closed and hereditary,

(2) $\mathcal{C}$ is cartesian closed and every $\mathcal{C}$–object $A$ has a *one–point–extension* $A^* \in |\mathcal{C}|$, i.e. every $A \in |\mathcal{C}|$ can be embedded via the addtion of a single point $\infty_A$ into a $\mathcal{C}$–object $A^* = A \cup \{\infty_A\}$ such that the following holds: For every $\mathcal{C}$–morphism $f : C \to A$ from a subspace $C$ of $B$ into $A$ the function $f^* : B \to A^*$ defined by

$$f^*(b) = \begin{cases} f(b) & , \text{ if } b \in C \\ \infty_A & , \text{ if } b \notin C \end{cases}$$

is a $\mathcal{C}$–morphism.

**2.13 Examples.** **SUConv**, **Fil** and **KConv** (= category of Kent convergence spaces) are topological universes (cf. [9;2.14.] for further examples).

## 3. Locally precompact semiuniform convergence spaces

**3.1 Definition.** A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *locally precompact* (resp. *locally compact*) provided that each $\mathcal{F} \in \mathcal{J}_X$ contains a precompact (resp. compact) subset of the product space $(X, \mathcal{J}_X) \times (X, \mathcal{J}_X)$.

**3.2 Proposition.** *Every locally compact semiuniform convergence space is locally precompact.*

**3.3 Proposition.** *Let $(X, \mathcal{J}_X)$ be a $t$–regular complete semiuniform convergence space. If $(X, \mathcal{J}_X)$ is locally precompact, then $(X, \mathcal{J}_X)$ is locally compact.*

**Proof.** Let $\mathcal{F} \in \mathcal{J}_X$. Since $(X, \mathcal{J}_X)$ is $t$–regular, $\overline{\mathcal{F}}^t \in \mathcal{J}_X$. By assumption, there is a precompact subset $H$ of $(X, \mathcal{J}_X) \times (X, \mathcal{J}_X)$ with $H \in \overline{\mathcal{F}}^t$. Thus, there is a closed subset $F$ of $X \times X$ with $F \in \mathcal{F}$ and $F \subset H$. Consequently, $F$ is precompact (cf. [9;4.2.]) and complete (cf. [9;3.12.1)]), i.e. $F$ is compact.

**3.4 Corollary.** *A $t$–regular convergence space (in particular a regular topological space) is locally precompact iff it is locally compact.*

**Proof.** Apply 3.2 and 3.3. Note that each convergence space is complete (cf. [9;3.7.]).

**3.5 Proposition.** 1) *Let $(X, \mathcal{J}_X)$ be a precompact semiuniform convergence space. Then $(X, \mathcal{J}_X)$ is locally precompact.*

2) *A diagonal semiuniform convergence space $(X, \mathcal{J}_X)$ is precompact provided that it is locally precompact.*

**Proof.** Analogously to [10;3.2.].

**3.6 Corollary.** *A principal uniform limit space (= uniform space) is precompact iff it is locally precompact.*

**3.7 Remark.** If a diagonal semiuniform convergence space $(X, \mathcal{J}_X)$ is locally compact, then it is locally precompact. The inverse implication is true provided that $(X, \mathcal{J}_X)$ is complete.

**3.8 Proposition.** *Let $(X, \mathcal{J}_X)$ be a locally precompact semiuniform convergence space. Then the underlying filter space $(X, \gamma_{\mathcal{J}_X})$ is locally precompact, i.e. each $\mathcal{F} \in \gamma_{\mathcal{J}_X}$ contains a precompact subset of $(X, \gamma_{\mathcal{J}_X})$.*

**Proof.** Let $\mathcal{F} \in \gamma_{\mathcal{J}_X}$. Then $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X$, and (by assumption) there is some precompact $H \in \mathcal{F} \times \mathcal{F}$, i.e. $H \supset F \times F$ for some $F \in \mathcal{F}$. Since the first projection $p_1 : (X, \mathcal{J}_X) \times (X, \mathcal{J}_X) \to (X, \mathcal{J}_X)$ is uniformly continuous and therefore Cauchy continuous, it follows from 2.11. that $p_1[H] \supset F$ is precompact and thus $F$ is precompact (cf. [9;4.3.]).

**3.9 Proposition.** *Let $(X,\gamma)$ be a filter space and $(X,\mathcal{J}_\gamma)$ its corresponding semiuniform convergence space, i.e. $\mathcal{J}_\gamma = \{\mathcal{F} \in F(X\times X)$: there is exists some $\mathcal{G} \in \gamma$ with $\mathcal{G}\times\mathcal{G} \subset \mathcal{F}\}$. Then the following are equivalent:*

(1) *$(X,\mathcal{J}_\gamma)$ is locally precompact.*

(2) *$(X,\gamma)$ is locally precompact, i.e. for each $\mathcal{F} \in \gamma$ there is some $F \in \mathcal{F}$ such that $F$ is a precompact subset of $(X,\gamma)$.*

**Proof.** (1) $\Rightarrow$ (2). Since $\gamma_{\mathcal{J}_\gamma} = \gamma$, this implication follows immediately from 3.8.

(2) $\Rightarrow$ (1). Let $\mathcal{G} \in \mathcal{J}_\gamma$. Then there exists some $\mathcal{F} \in \gamma$ with $\mathcal{F} \times \mathcal{F} \subset \mathcal{G}$. By assumption, there is some precompact $F \in \mathcal{F}$. Thus, $F \times F$ is precompact (cf. [9;4.3.]) and belongs to $\mathcal{G}$.

**3.10 Proposition.** *The category* **LPC–SUConv** *of locally precompact semiuniform convergence spaces (and uniformly continuous maps) is a bicoreflective subcategory of* **SUConv**, *where $1_X : (X,\mathcal{J}_X^*) \to (X,\mathcal{J}_X)$ is the bicoreflection of $(X,\mathcal{J}_X) \in |$***SUConv***$|$ with respect to* **LPC–SUConv** *provided that $\mathcal{J}_X^* = \{\mathcal{F} \in \mathcal{J}_X$: there is some $F \in \mathcal{F}$ such that $F$ is a precompact subset of $(X,\mathcal{J}_X) \times (X,\mathcal{J}_X)\}$.*

**Proof.** Analogously to [10;3.6.].

**3.11 Proposition.** **LPC–SUConv** *is closed under formation of finite products and subspaces (in* **SUConv***).*

**Proof.** Cf. [10;3.8.] and note that in contrast to the situation for compact spaces all subspaces (not only the closed ones) of precompact semiuniform convergence spaces are precompact.

**3.12 Theorem.** **LPC–SUConv** *is a topological universe.*

**Proof.** 1) The fact that **LPC–SUConv** is a cartesian closed topological category is proved analogously to [10;3.9.].

2) Since **SUConv** is hereditary and **LPC–SUConv** is a bicoreflective subcategory of **SUConv** which is closed under formation of subspaces in **SUConv**, it follows from [12;4.1.] that **LPC–SUConv** is hereditary.

**3.13 Remarks.** 1) The power–objects in **LPC–SUConv** are formed as follows: Let $\mathbf{X} = (X,\mathcal{J}_X)$ and $\mathbf{Y} = (Y,\mathcal{J}_Y)$ be locally precompact semiuniform convergence spaces; then the *power–object* $\mathbf{Y}^{\mathbf{X}}$ is the set $[\mathbf{X},\mathbf{Y}]$ of all uniformly continuous maps from $\mathbf{X}$ into $\mathbf{Y}$ endowed with the **LPC–SUConv**–structure $\mathcal{J}_{X,Y}^* = \{\phi \in \mathcal{J}_{X,Y}$: there is some $K \in \phi$ such that $K$ is a precompact subset of $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y}) \times ([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$, where $J_{X,Y} = \{\phi \in F([\mathbf{X},\mathbf{Y}] \times [\mathbf{X},\mathbf{Y}]) : \phi(\mathcal{F}) \in \mathcal{J}_Y$ for each $\mathcal{F} \in \mathcal{J}_X\}$ and $\phi(\mathcal{F})$ is the filter generated by $\{A(F) : A \in \phi, F \in \mathcal{F}\}$ with $A(F) = \{(f(a),g(b)) : (f,g) \in A, (a,b) \in F\}$ (cf. [9;7.1.1)] and [11;5.1.(2)]).

2) The one–point extensions in **LPC–SUConv** are formed as follows: Let $(X, \mathcal{J}_X)$ be a locally precompact semiuniform convergence space and $X^* = X \cup \{\infty_X\}$ with $\infty_X \notin X$; then the *one–point extension* of $(X, \mathcal{J}_X)$ in **LPC–SUConv** is $(X^*, (\mathcal{J}_X^{\#})^*)$, where $(\mathcal{J}_X^{\#})^* = \{\mathcal{F} \in \mathcal{J}_X^{\#}$: there is some $F \in \mathcal{F}$ such that $F$ is a precompact subset of $(X^*, J_X^{\#}) \times (X^*, J_X^{\#})\}$ and $J_X^{\#}$ denotes the **SUConv**–structure on $X^*$ generated by $\{\mathcal{G}^{\#} : \mathcal{G} \in \mathcal{J}_X\}$ with $\mathcal{G}^{\#} = \{G^{\#} : G \in \mathcal{G}\}$ provided that $G^{\#} = G \cup (X^* \times \{\infty_X\}) \cup (\{\infty_X\} \times X^*)$ (cf. [2;1.5.8] and [12;4.1.])

**3.14 Proposition.** *The object class of the bicoreflective hull (formed in* **SUConv***) of all precompact semiuniform convergence spaces, i.e. the class of all* **precompactly generated semiuniform convergence spaces***, consists exactly of all semiuniform convergence spaces* $(X, \mathcal{J}_X)$ *for which* $\mathcal{J}_X$ *is the final* **SUConv***–structure with respect to the family* $(j_i : (K_i, \mathcal{J}_{K_i}) \to (X, \mathcal{J}_X))$ *of the inclusions of all precompact subspaces of* $(X, \mathcal{J}_X)$.

**Proof.** Analogously to [10;3.11].

**3.15 Theorem.** *A semiuniform convergence space* $(X, \mathcal{J}_X)$ *is precompactly generated if and only if it is locally precompact.*

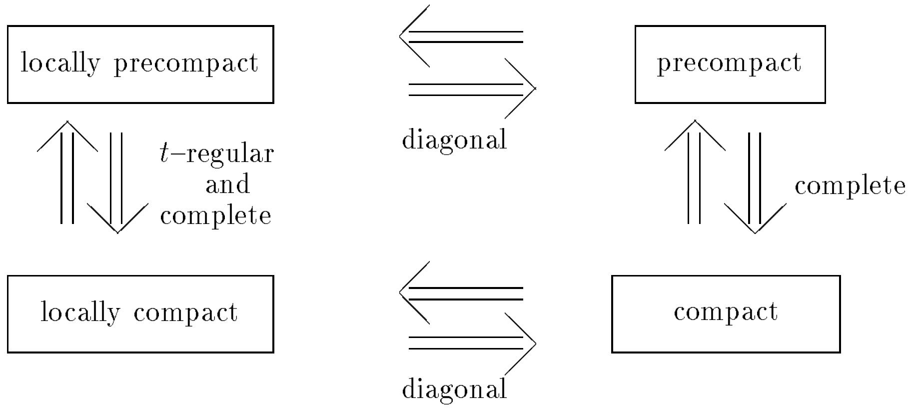
**Proof.** Cf. the corresponding proof of 3.12. in [10].

## 4. Final remarks

**4.1.** The category **LPC–Fil** of locally precompact filter spaces (and Cauchy continuous maps) can be embedded into **LPC–SUConv** as a bicoreflective subcategory (namely: if $C$ : **SUConv** $\to$ **Fil–D–SUConv** ($\cong$ **Fil**) is the bicoreflector assigning to each semiuniform convergence space its underlying filter space, then using 3.8. the restriction of $C$ to **LPC–SUConv** is the desired bicoreflector). Since by 3.10. **LPC–SUConv** is a bicoreflective subcategory of **SUConv**, **LPC–Fil** can also be embedded into **SUConv** as a bicoreflective subcategory. By 3.11. and [8;3.8.], **LPC–Fil** is closed under formation of finite products and subspaces (both formed in **SUConv**). Consequently, **LPC–Fil** *is a topological universe* (use 2.13., [6;2.4.], and [12;4.1.]). This one has been used by H.L. Bentley and H. Herrlich [3] for proving two *theorems of the Ascoli type*, where **Fil** is replaced by the isomorphic category **Grill** of grill–determined prenearness spaces.

**4.2.** Local precompactness in uniform limit spaces plays an essential role in Wyler's *Ascoli theorem* [14;4.5.], where, according to 3.15., 'local precompact' means 'precompactly generated'. Wyler's version of the Ascoli theorem is a generalization of Bourbaki's Ascoli theorem [4;§2, no. 5, théorème 2].

**4.3.** The relationships between compactness, precompactness, local compactness and local precompactness can be summarized by means of the following diagram (cf. also [9;4.4. and 4.5.]):




## References


1. J. Adámek, H. Herrlich and G.E. Strecker, Abstract and Concrete Categories, (Wiley–Intersci. Publ., New York, 1990).
2. A. Behling, Einbettung uniformer Räume in topologische Universen, Doctoral Dissertation, Free University, Berlin (1992).
3. H.L. Bentley and H. Herrlich, Ascoli's theorem for a class of merotopic spaces, in: Proc. Conf. Bechyne 1984, Math. Research **24** (Akademie Verlag, Berlin, 1985) 47–53.
4. N. Bourbaki, Topologie générale, chap. 10 (Hermann, Paris, 1961).
5. V.A. Efremovič, Geometry of proximity, Mat. Sb. 31 (73) (1952), 189–200.
6. L.D. Nel, Initially structured categories and cartesian closedness, Canad. J. Math. 27 (1975), 1361–1377.
7. G. Preuß, Theory of Topological Structures (Reidel, Dordrecht 1988).
8. G. Preuß, Cauchy spaces and generalizations, Math. Japonica 38 (1993), 803–812.
9. G. Preuß, Semiuniform convergence spaces, Math. Japonica 41 (1995), 465–491.
10. G. Preuß, Local compactness in semiuniform convergence spaces, Quaest. Math. (to appear).
11. F. Schwarz, Product compatible reflectors and exponentiability, in: Categorical Topology, Sigma Series in Pure Mathematics 5 (Heldermann, Berlin, 1984) 505–522.
12. F. Schwarz, Hereditary topological categories and topological universes, Quaest. Math. 10 (1986), 197–216.
13. A. Weil, Sur les espaces à structures uniformes et sur la topologie générale (Hermann, Paris, 1937).
14. O. Wyler, Ascoli theorems for cartesian closed categories, in: Categorical Topology, Sigma Series in Pure Mathematics 5 (Heldermann, Berlin, 1984) 599-617.