On the mathematical background of Gilbreath's card trick

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Abstract. In Martin Gardner's book "New mathematical diversions from Scientific American" it is described how certain information concerning the initial distribution of a deck of cards survives an arbitrarily thorough riffle shuffle. This gives rise to a card trick which is attributed to Norman Gilbreath, several other tricks presented in magic journals are based on the same principle.

Here we aim to generalize the underlying ideas. We consider sequences $a_1, a_2, \ldots$ with values in $\{-1, +1\}$ for which the sequence $a_1, a_1+a_2, \ldots$ of the associated partial sums only has small oscillations. It is indicated how general properties of such (a_n) can be used to understand what happens when shuffling cards and how this can be given the form of a card trick. The mathematics involved is rather elementary.

1. Introduction

Gardner's version of the fact which underlies the *Gilbreath card trick* reads as follows (cf. chapter 9 of [4]):

> *Consider a deck of $2n$ cards ordered in such a way that red and black cards alternate. Cut the deck into two parts and give it a riffle shuffle. Cut it once more, this time not completely arbitrarily but at a place where two cards of the same colour meet. Square up the deck.*
>
> *Then for every $i = 1, \ldots, n$ the pair consisting of the $(2i-1)$-th and the $2i$-th card is of the form* (red,black) *or* (black,red).

Despite of the mathematical flavour of this trick there seem to be no investigations which treat the mathematical background. It appears only in [1] where a computer-based proof is presented. On the other hand, from the probabilistic point of view shuffling of cards has attracted the attention of many authors, the interested reader is referred to [2],[3],[5],[6] and the references cited there. These investigations, however, have little impact on card tricks since there – for obvious reasons – information which is secure is needed.

Here is the definition which will be crucial for this paper:

Definition 1.1 Let k, m be integers. A sequence $A = (a_1, \ldots, a_m)$ with values in $\{-1, +1\}$ will be called *k-bounded* provided that $|a_r + \cdots + a_s| \leq k$ for arbitrary integers r, s with $1 \leq r \leq s \leq m$. A is said to be *properly k-bounded* if it is k-bounded but not $(k-1)$-bounded.

Sequences in $\{-1, +1\}$ are of interest in connection with the Gilbreath card trick since with the obvious identification *"red"* $= -1$, *"black"* $= +1$ decks of cards correspond to certain sequences in $\{-1, +1\}$, and it will be shown how k-boundedness comes into play in a natural way. Note that in view of this correspondence the notion "a k-bounded deck" is well-defined.

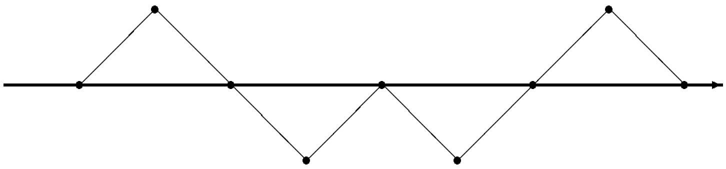
We will see in *section 2* how k-boundedness is connected with shuffling. Also we investigate what can be said about k-bounded sequences certain elements of which are known. It will turn out that in many cases poor information suffices to draw conclusions. In *section 3* we present a short guide to disguise the mathematical facts as card tricks.

2. Sequences which are k-bounded

Let $a_1, \ldots, a_m$ be k-bounded. Clearly $a_m, a_{m-1}, \ldots, a_2, a_1$ will be k-bounded as well, and $a_{r+1}, a_{r+2}, \ldots, a_m, a_1, \ldots, a_r$ (with $1 \leq r \leq m$) will also have this property provided that $a_1 + \cdots + a_m = 0$. In the translation of card decks this means that k-boundedness is preserved by reversing the order of the cards as well as by the operation "cut the deck arbitrarily and square it up" (the latter under the additional assumption that the number of red cards equals the number of black ones).

k-boundedness *can easily be visualized.* Given $a_1, \ldots, a_m$, consider the points $(0,0), (1, a_1), (2, a_1 + a_2), \ldots, (m, a_1 + \ldots + a_m)$ in the plane and join two consecutive points by a line segment; this will be called the *graph* of $a_1, \ldots, a_m$. Then the sequence is k-bounded iff the width of its graph (i. e. the difference between the maximum and the minimum of the sequence $0, a_1, a_1 + a_2, \ldots, a_1 + \ldots + a_m$) is at most k.

Here is the graph of the 2-bounded sequence $+1, -1, -1, +1, -1, +1, +1, -1$:



Now suppose that $A := (a_1, \ldots, a_m)$ and $B := (b_1, \ldots, b_{m'})$ are k-bounded and k'-bounded, respectively. Imagine two piles of cards such that the sequence of red and black cards in the first (resp. the second) pile is precisely the sequence of the -1's and $+1$'s in A (resp. B). Now give these decks a riffle shuffle and translate the result into a sequence C of -1's and $+1$'s. Sequences C which arise in this way will be called a *shuffle mixture of A and B.*

> We hasten to add that, of course, a precise definition is also possible: C is a shuffle mixture of A, B iff there are partitions $i_1 < \cdots < i_r$ of $1, \ldots, m$ and $j_1 < \cdots < j_{r'}$ of $1, \ldots, m'$ such that $\ldots$; the reader should be able to fill in the details.

Here is our key observation:

Proposition 2.1 *Let C be a shuffle mixture of A and B which are k- and k'-bounded, respectively. Then C is $(k + k')$-bounded.*

Proof. This follows immediately from the triangle inequality, since every sum of the form $c_{r''} + c_{r''+1} + \cdots + c_{s''}$ with $r'' \leq s''$ can be written as $a_r + a_{r+1} + \cdots + a_s + b_{r'} + b_{r'+1} + \cdots + b_{s'}$ with suitable $r \leq s$, $r' \leq s'$. The result is also clear from inspection of the associated graphs. □

This proposition gives rise to some assertions concerning riffle shuffles of cards:

- Suppose one shuffles a 1-bounded deck of cards as in the Gilbreath card trick described above. Then the result will be 2-bounded.
- Divide a 1-bounded deck into three piles. Give the first two of these a riffle shuffle and then repeat this with the result of the first shuffle and the remainig third pile. In this way one obtains a 3-bounded deck.
- A 4-bounded deck results from a 1-bounded one by applying two times the above Gilbreath shuffling procedure.

- Suppose that a deck of cards contains k red as well as k black cards. There exists a permutation which is 1-bounded and one which is k-bounded but *not* $(k-1)$-bounded. Since the first permutation transforms to a 2^r-bounded deck after r riffle shuffles à la Gilbreath it follows that r such shuffles will suffice to generate every permutation of the cards only if $2^r \geq k$.

More generally it is possible by these procedures to provide k-bounded decks also for other k. Note, however, that a k-bounded sequence might be $(k-1)$-bounded. When sequences are generated by riffle shuffles of cards this might happen if k is too large or if the person who does the job is unable to shuffle properly. We will return to this point since this might cause some problems when dealing with card tricks.

What can be said about a deck of cards once one knows that it is k-bounded? A trivial fact, for example, is that any consecutive $k+1$ cards contain both red and black cards. In order to derive less obvious properties we will need the following definition.

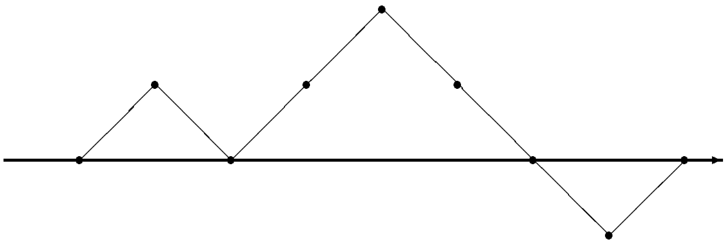
Definition 2.2 Let $A = (a_1, \ldots, a_m)$ be k-bounded. We denote by H_A the minimum of the sequence $\alpha_0 := 0$, $\alpha_1 := a_1$, $\alpha_2 := a_1 + a_2, \ldots \alpha_m := a_1 + \cdots + a_m$.

The *height of the i-th element* is then defined as the number $h_i^A := \alpha_i - H_A$ for $i = 0, \ldots, m$.

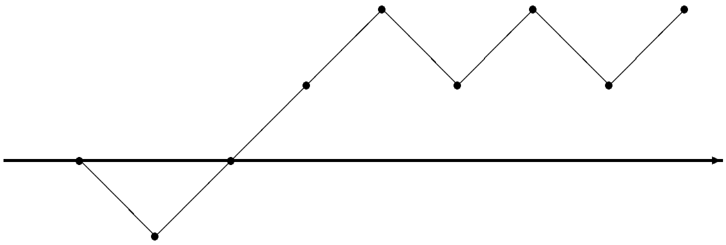
(If A is clear from the context we will write h_i instead of h_i^A.)

Clearly the h_i are in $\{0, \ldots, k\}$, and all these numbers can easily be visualized once the graph of $a_1, \ldots, a_m$ has been sketched. For example, for the 2-bounded sequence $+1, -1, -1, +1, -1, +1, +1, -1$ (for which we have indicated the graph above) the $h_0, \ldots, h_8$ are $1, 2, 1, 0, 1, 0, 1, 2, 1$.

Let us consider an *example* to illustrate how one can derive information from the h_i. Let a 3-bounded sequence be given which is not 2-bounded and suppose that for some reason one knows that $h_0 = 1$. The associated graph might look as follows:



Or like this:



Nothing can be said about a_1 in this case, but something is known about the pair (a_1, a_2): This pair will *not* be $(-1, -1)$, and no other assertions can be made in general. What about (a_1, a_2, a_3)? Both -1 and $+1$ occur in this triple, $(-1, -1, +1)$ is not possible, and nothing more can be guaranteed.

Translating this fact into properties of cards we know: If a 3-bounded deck of cards satisfies $h_0 = 1$, then the first triple will show at least one red and one black card.

We now turn to the *general case*. Motivated by the preceding example we are faced with *two problems*:

- What can be said about a k-bounded sequence if some h_{i_0} is known?

- How can one determine some h_{i_0} with as little information as possible?

In order to have an appropriate notation when dealing with the first problem we introduce the following definition; it can be thought of as a deterministic variant of conditional probabilities.

Definition 2.3 Denote by $S_m^k \subset \{-1,+1\}^m$ the collection of k-bounded sequences of length m. Let π be a property the $A \in S_m^k$ can have. Then, for integers i_0, r such that $0 \le i_0 < i_0+r \le m$ we denote by $I_m^k(a_{i_0+1},\ldots,a_{i_0+r}|\pi)$ $\subset \{-1,+1\}^r$ the collection of all $(a_{i_0+1},\ldots,a_{i_0+r})$ such that $(a_1,\ldots,a_m)$ is in S_m^k and satisfies π.

To phrase it otherwise: $I_m^k(a_{i_0+1},\ldots,a_{i_0+r}|\pi)$ is the best possible information on $a_{i_0+1},\ldots,a_{i_0+r}$ provided one knows that π holds.

With this notation the preceding facts are just the statements

$$\begin{aligned} I_m^3(a_1|h_0=1) &= \{-1,+1\},\\ I_m^3(a_1,a_2|h_0=1) &= \{(-1,+1),(+1,-1),(+1,+1)\},\\ I_m^3(a_1,a_2,a_3|h_0=1) &= \{(-1,+1,-1),(-1,+1,+1),(+1,-1,-1),\\ &\qquad (+1,-1,+1),(+1,+1,-1)\}. \end{aligned}$$

And here are some further examples:

$$\begin{aligned} I_m^k(a_3|h_2=0) &= \{+1\},\\ I_m^k(a_3,a_4|h_2=k) &= \{(-1,-1),(-1,+1)\},\\ I_m^5(a_1,\ldots,a_5|h_0=0,h_5=5) &= \{(+1,+1,+1,+1,+1)\},\\ I_m^3(a_1,a_2|h_0=1,a_3=+1) &= \{(-1,+1),(+1,-1)\}. \end{aligned}$$

In any case the I_m^k can easily be determined if one visualizes k-bounded sequences by using their graphs.

To prepare the discussion of the *second problem* we treat in some detail *the case* $k=2$. Suppose we know about a sequence that it is 2-bounded, we assume for the moment that it is *not* 1-bounded, and we want to determine h_{i_0} for a suitable index i_0. We describe two possibilities:

- Let r be an index with $1 \le r < m$, $a_r = a_{r+1}$; then $h_r = 1$ (more generally: $h_{r'} = 1$ for all $0 \le r' \le m$ with $r \bmod 2 = r' \bmod 2$).
- Suppose that our sequence – call it C – is a shuffle mixture of two 1-bounded sequences A, B for which $h_0^A, h_0^B \in \{0,1\}$ are known. Then $h_0^C = h_0^A + h_0^B$; this is rather obvious if one considers the associated graphs, a more general statement can be found in Proposition 2.5.

These facts give rise to variants of the Gilbreath card trick (see the next section), the generalizations to the case of arbitrary k read as follows:

Proposition 2.4 *Suppose that one knows a subsegment* $B = (a_{i_0+1}, \ldots, a_{i_0+r})$ *of a sequence* A *which is known to be* k*-bounded. If* B *is properly* k*-bounded, then all* $h^A_{i_0+1}, \ldots, h^A_{i_0+r}$ *are known.*

Proposition 2.5 *Let* C *be a shuffle mixture of a* k*-bounded* A *and a* k'*-bounded* B *(so that* C *is* $(k+k')$*-bounded by Proposition 2.1). If* C *is* properly $(k+k')$*-bounded, then* $h^C_0 = h^A_0 + h^B_0$.

Proof. Les us denote the sequence of partial sums for A, B, C by $(\alpha_i)_{i=0,\ldots}$, $(\beta_i)_{i=0,\ldots}$, $(\gamma_i)_{i=0,\ldots}$, by H_A, H_B, H_C their minima and H^*_A, H^*_B, H^*_C their maxima, respectively. Then, by the definition of shuffle mixture, every γ_i is of the form $\alpha_{i'} + \beta_{i''}$ for suitable i', i'', hence $H_C \geq H_A + H_B$, $H^*_C \leq H^*_A + H^*_B$. But $H^*_A - H_A \leq k$, $H^*_B - H_B \leq k'$ and $H^*_C - H_C = k + k'$ so that all inequality signs can be replaced by " $=$ ". In particular, $H_C = H_A + H_B$ holds, and this proves our claim. □

It seems to be hopeless to give a complete catalogue of how to draw conclusions from certain information on k-bounded decks/sequences. The reader is invited to find further statements, we now turn to the translation into card tricks.

3. Applications to card tricks (or: the honest magician)

Let us first discuss the ideas when dealing with the Gilbreath trick described at the beginning: We start with a 1-bounded deck containing an even number of cards, and someone produces a shuffle mixture of an initial segment and the remaining cards. This will be (not necessarily properly) 2-bounded. We propose *three variants* to find an i_0 for which h_{i_0} is known:

- Let anyone from your audience cut the deck at a position where two cards of the same colour meet and square it up. Then the resulting deck is properly 2-bounded, $h_0 = 1$ holds so that every pair of the form (a_{2i-1}, a_{2i}) has two different elements.

  (It should be clear that this raw fact needs some additional action to be considered as a remarkable card trick: Hide the deck and pretend to work extremely hard to find pairs containing both colours etc.)

This is the original variant of the trick. Note that the step "cut it at a place where two cards of the same colour meet" can only be done if the riffle shuffle produced a deck which is *properly* 2-bounded. Hence one should choose a person which has some experience with this kind of shuffling, in this case the probability of getting a 1-bounded deck is extremely small (see the discussion at the end of this paper).

- (The following idea is presented in [1].) After the deck was cut for the first time one has a look at the bottom cards of the two piles. Suppose they are of the same colour, this is equivalent with the fact that both parts contain an even number of cards. We claim that after the riffle shuffle every pair *(card number $2i$,card number $2i+1$)* shows a black as well as a red card. This is trivial if the deck is 1-bounded, and if it is properly 2-bounded the assertion follows from Proposition 2.5: By assumption the decks A, B which arise after the first cut satisfy $h_0^A = h_0^B$ so that $h_0 \in \{0, 2\}$ for the shuffled deck; in any case, $h_1 = 1$ and this implies the assertion.

  One can argue similarly if the bottom cards show different colours. Then $h_0 = 1$, and the pairs *($(2i-1)$-th card,$(2i)$-th card)* will show both colours.

- In this last variant we apply Proposition 2.3: Take the deck and switch one card after the other until – after the r-th card, say – for the first time two consecutive cards are of the same colour. Then $h_{r-1} = 1$ and you have all the information you need for the remaining cards.

It is not hard to add some *refinements*. Suppose you start with a 1-bounded deck containing an even number of cards. Denote by $\mathcal{C}$ (resp. $\mathcal{C}_{even}/\mathcal{C}_{odd}$) the following operation: Cut the deck arbitrarily (resp. into two parts containing both an even/odd number of cards) and square it up. Then you may let apply $\mathcal{C}$ as often as you wish, it will not destroy 1-boundedness. Then you get a 2-bounded deck by the usual procedure, and again $\mathcal{C}$ might be applied. The next step is to determine an i_0 with $h_{i_0} = 1$ by one of the variants described above.

At this stage $\mathcal{C}$ would possibly destroy the information. However, $\mathcal{C}_{even}$ or $\mathcal{C}_{odd} \circ \mathcal{C}_{odd}$ will lead to a deck where $h_{i_0} = 1$ for the same i_0 as before.

Here is a short guide to use similar ideas in *the case of general k-bounded decks.*

Step 1: Prepare a 1-bounded deck containing an even number of cards. Ask someone to shuffle it in such a way that a k-bounded deck results; how to do this for an arbitrary k is described after Proposition 2.1. Before and after this procedure you or anybody else might apply the $\mathcal{C}$-operation arbitrarily often.

Step 2: This is the critical point: Try to find h_{i_0} for a "small" i_0. As in the case $k = 2$ there are (at least) the following three possibilities:

- Ask someone to cut the deck at a position where k cards of the same colour meet. If the deck is cut after the first of these cards then after squaring it up you know that $h_0 = 1$ or $= k - 1$ depending on the colour of these cards. The situation is more favourable if k is even since then you can be sure that a cut after the $k/2$-th card will yield $h_0 = k/2$.

- You can try to apply Proposition 2.5. This necessitates of course that you know the h_0-values of the two piles before the last riffle shuffle, but this can be done by means of the preceding or the following possibility (applied to both piles).

  Note that the proposition can only be applied if the final deck is properly k-bounded.

- Switch – one after the other – the first r cards from the top such that r is as small as possible and $a_1, \ldots, a_r$ is properly k-bounded. Then, for the remaining deck, h_0 will be 0 or k according to whether the last card was red or black.

Step 3: Suppose for simplicity that you know h_0. It is described after Definition 2.3 how this provides information.

Suppose e. g. that $k = 4$ and $h_0 = 1$. You will have many possibilities: Any subsegment of the form $a_{4i-3}, a_{4i-2}, a_{4i-1}, a_{4i}$ will contain cards of different colours since h_{4i} is either 1 or 3 for each i. But more is true. You know the precise value of h_4 after you have seen the first four cards, h_8 is known after $a_5, \ldots, a_8$ are switched, and so on. This allows you to refine your prediction in that not only "both colours occur among the next four cards" is true but even "both colours, but at least as many black as red ones" (if $h_0 = 1$ holds).

After each prediction and presentation of cards you can update your knowledge so that at every stage you know what h_0 is for the remaining deck.

Let's now turn to *probabilities*. At least two questions are of interest in connection with card tricks, namely:

- What is the probability that a k-bounded deck with m cards is properly k-bounded? If this is not extremely small it might be difficult to achieve step 2.
- For a properly k-bounded deck containing m cards let $r^* \in \{k, \ldots, m\}$ be the smallest r such that $a_1, \ldots, a_r$ is properly k-bounded. What is the expected value of r^*? It has to be small compared with m if you want to present the third variant of step 3 convincingly.

In order to treat such questions properly one needs a probability measure P on the collection of all k-bounded sequences of length m. It is by no means canonical what P should be chosen, it heavily depends on the way how the riffle shuffle is done. Let's take for simplicity the uniform distribution. Even in this case it seems to be difficult to calculate exact probabilities in the case of reasonably large m. Here are the results of some computer simulations for the case of 32 cards (16 black and 16 red ones) and some small k:

- Practically all 2-bounded sequences are properly 2-bounded. This is true even for a deck of 16 cards. With 8 cards one gets a properly 2-bounded deck among the 2-bounded ones with a probability of roughly 0.87.

  The expected value of r^* is independent of m for reasonably large m and it is rather small, it is about 4.4. Thus – in the average – after having seen 5 cards of the deck you can pass to step 3.

- The situation with $k = 3$ (resp. $k = 4$) is less favourable. Only about 95% (resp. 80%) will be proper.

  The expected value of r^* is about 10 (resp. 19).

Finally it should be noted that this is not the end of the story. One has a similar theory if the sequence under consideration has its values not necessarily in $\{-1, +1\}$ but can attain arbitrary nonzero integers. With the same definition of k-boundedness it is possible to generalize most of the results, the formulations, however, will sometimes look rather clumsy.

The application to card tricks is obvious, you will be able to predict more than merely the colour. (As an example assign the following values: Every red card counts -1 and every black one $+1$ as before, but if the card does *not* show a number then its value is doubled. Prepare a 2-bounded deck and continue as above. Then, after you ended up with a 4-bounded deck for which you know that $h_0 = 1$ you can guarantee that the next card is *not* a black queen.)

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