SERIE A — MATHEMATIK

9-connected claw-free graphs are Hamilton-connected

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Preprint No. A 97–15
August 1997

Abstract

A graph is Hamilton-connected if any pair of vertices is joined by a hamiltonian path. In this note it is shown that 9-connected graphs which contain no induced claw $K_{1,3}$ are Hamilton-connected, by reformulating and generalizing a closure concept due to Ryjáček, which turns claw-free graphs into line graphs.

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Introduction and main result

Any undefined notation and concepts can be found in [9]. The famous 2-tough conjecture, going back to Chvátal, says that every 2-tough graph is hamiltonian. Restricted to claw-free graphs it says that 4-connected claw-free graphs are hamiltonian, since it was shown in [7] that the toughness of a claw-free graph is exactly half its connectivity.

In November 1995, the "EIDMA-Workshop on the hamiltonicity of 2-tough graphs" took place in the Netherlands near Enschede, stimulating two striking developments: the 2-tough conjecture itself was refuted by Bauer, Broersma and Veldman [1], by constructing non-hamiltonian graphs with toughness arbitrarily close to 9/4, while further evidence for its restriction to claw-free graphs was given by Ryjáček [8], by showing that 7-connected (and therefore 7/2-tough) claw-free graphs are hamiltonian. No constant connectivity bound guaranteeing hamiltonicity of a claw-free graph was previously known.

For this purpose, Ryjáček developed a closure concept for claw-free graphs, which leaves the lengths of longest cycles [8] and paths [3] unchanged, and which turns the graph into the line graph of a triangle-free graph. Since Zhan [10] proved that 7-connected line graphs are Hamilton-connected (so, in particular, hamiltonian), Ryjáček's closure concept implies that 7-connected claw-free graphs are hamiltonian. Unfortunately, the length of longest paths joining two specified vertices can change in the closure process (see [3] for an example), so Ryjáček's closure combined with Zhan's result does not imply that 7-connected claw-free graphs are Hamilton-connected.

In [3] it was conjectured that there is a constant k such that the property of being Hamilton-connected becomes stable for k-connected claw-free graphs under Ryjáček's closure. The main result of this note is to verify the conjecture for $k = 9$.

Theorem 1 *Every 9-connected claw-free graph is Hamilton-connected.*

This trivially implies that the property "Hamilton-connected" is stable for 9-connected claw-free graphs. As a tool, we will give a different interpretation of Ryjáček's closure concept which is somewhat easier to handle, and then restrict the closure operation to certain parts of the graph to preserve Hamilton-connectivity.

A different view on Ryjáček's closure

Closure concepts in hamiltonian graph theory were introduced by Bondy and Chvátal in 1976 [2], unifying and generalizing many results in this subject. Since then many variants were considered, the last and very powerful one by Ryjáček [8] for claw-free graphs.

Ryjáček's closure of a claw-free graph G in its original form consists in adding all missing edges to the neighborhood of a vertex which forms a connected graph. This operation preserves claw-freeness and the length of a longest cycle. Continuing with this operation until the neighborhood of every vertex is disconnected or complete, we end with a unique graph $\mathrm{cl}_R(G)$, which is the line graph of a triangle-free graph.

The equivalent interpretation is:
Successively add the missing edge to an induced $K_4 - e$ until the graph is $K_4 - e$-free. The final result is uniquely determined and again claw-free, while it may be unavoidable to obtain induced claws in the closure process. Moreover it is $K_4 - e$-free and hence the line graph of a triangle-free graph (see e.g. [6, Theorem 5.2]).

The operation of filling in the missing edge into induced $K_4 - e$'s was proposed by Broersma [4], who also observed that this operation applied once to a claw-free graph preserves the length of a longest cycle while claw-freeness may be lost. Broersma extracted a neighborhood condition that makes the closure operation leave the length of a longest cycle unchanged, turning it into a meaningful closure concept for general graphs. Nevertheless, this so-called K_4-closure applied to claw-free graphs is weaker than Ryjáček's closure and, in general, not unique. Recently, Broersma and Trommel [5] developed generalizations of the K_4-closure based on neighborhood requirements, which leave the lengths of a longest cycle unchanged and create a supergraph of the Ryjáček closure. These closures are still not unique and the formulations are more technical.

It was shown that for claw-free graphs G the closure $\mathrm{cl}_R(G)$ leaves the length of longest cycles [8] and of longest paths [3] unchanged. Conversely, it was shown in [3] that properties with a local appeal like pancyclicity are not stable even within claw-free graphs with large connectivity under Ryjáček's closure. Also claw-free graphs of connectivity 2 were found, which are not Hamilton-connected, while their closure is Hamilton-connected [3], and it was conjectured that this property becomes stable for claw-free graphs of large connectivity.

Here an affirmative answer is presented by showing that any 9-connected claw-free graph is Hamilton-connected, so the property becomes trivially stable for 9-connected claw-free graphs. The method is to apply the closure operation only to specific parts of the graph, for which we can show that the length of a longest path joining two given vertices is unchanged.

In $K_4 - e$ we call the vertices of degree 3 the center vertices. For a graph G and a subset $S \subseteq V(G)$, denote by $\mathrm{cl}_R(G, S)$ the supergraph of G which is obtained by successively adding the missing edges to induced $K_4 - e$'s, both center vertices of which are not in S. This concept generalizes Ryjáček's closure $\mathrm{cl}_R(G) = \mathrm{cl}_R(G, \emptyset)$. Let us first show, that the graph $\mathrm{cl}_R(G, S)$ is uniquely determined.

Theorem 2 *Let G be a graph and $S \subseteq V(G)$, then the closure $\mathrm{cl}_R(G, S)$ is uniquely determined.*

Proof. Assume that there are different closures G_1, obtained by adding the edges $e_1, e_2, \ldots, e_s$, and G_2, obtained by adding the edges $f_1, f_2, \ldots, f_t$ to G (in the given order). We may assume that G_1 contains an edge which is not contained in G_2. Let e_i be the edge in $E(G_1) \setminus E(G_2)$ which was first added to G_1. Then the endvertices of e_i are contained in an induced $K_4 - e$ in G_2 which has its center vertices in $V(G) \setminus S$. So G_2 is not closed, contradicting the assumption. $\square$

If we apply the closure operation to a graph G, in which all induced claws are centered in S, then in $\mathrm{cl}_R(G,S)$ all induced claws are centered in S. Instead of proving the general case, we now restrict to the case that we are really interested in, namely when S consists of two vertices a and b. We will simultaneously show that the length of a longest (a,b)-path in $\mathrm{cl}_R(G,\{a,b\})$ remains unchanged, and that $\mathrm{cl}_R(G,\{a,b\})$ has all its induced claws centered at a and b. To this goal, we come back to Ryjáček's idea of filling up the neighborhood of a specific vertex first before we proceed to the neighborhood of the next vertex. For a graph G let $G(x,\{a,b\})$ be the graph obtained from G by consecutively adding the missing edge into induced K_4-e's whose center is x and a vertex distinct from a,b.

Lemma 1 *Let G be a graph in which all induced claws are centered in two vertices a and b. Then the length of a longest (a,b)-path is unchanged in $G(x,\{a,b\})$, and all induced claws of $G(x,\{a,b\})$ are centered in a or b.*

Proof. In order to symmetrize the problem, we add an auxiliary edge $e^*=ab$ to obtain a graph G^*, which clearly has all its induced claws centered at a or b, and consider longest cycles through e^* instead of longest (a,b)-paths. Let us first show that every induced claw of $G'=G^*(x,\{a,b\})$ has its center in a or in b. Since G^* had this property, every induced claw of G' which is not centered in a or b has exactly one edge uw in $N(x)$. It cannot have more edges in $N(x)$, since this would result in an induced K_4-e whose center is x and a vertex different from a,b. Together with the two edges uz_1,uz_2 of the claw distinct from uw, the edge ux forms an induced claw in G^*, a contradiction.

Now assume that there is a longer (a,b)-path in G' than in G^*. Then there are graphs H,H' which are subgraphs of G' and supergraphs of G^*, with the property that H' is obtained from H by adding the missing edge $e=uv$ into an induced K_4-e centered at x and a vertex y different from a,b, such that a longest (a,b)-path in H' exceeds the length of a longest (a,b)-path in H.

Since in G^* all induced claws are centered in a or b, all induced claws of H centered in other vertices have an edge in the neighborhood of x. Let C be a longest cycle through e^* in H'. Then C contains e. So H contains the (u,v)-path $P=C-e$. By symmetry we may assume that P has vertices $v_1,v_2,\dots,v_p$ in cyclic order with $u=v_1,v=v_p,x=v_i,y=v_j$ such that $i<j$. Now the edge $v_pv_{i-1}\notin E(H)$, otherwise we obtain a cycle $v_1,v_2,\dots,v_{i-1},v_p,v_{p-1},\dots,v_i,v_1$ in H containing all edges of P except one edge incident with $x=v_i$, so, in particular, containing e^*. Similarly, $v_{i-1}v_{i+1},v_{j-1}v_{j+1}\notin E(H)$. This implies that $v_{i+1}v_p\in E(H)$ to avoid a claw centered in $x=v_i$ (see Figure 1). Now consider the graph induced by $\{v_j,v_{j-1},v_{j+1},v_i\}$. This cannot be an induced claw centered at v_j, since this claw would have no edge in $N(x)$. So one of the edges v_iv_{j-1},v_iv_{j+1} is present in H. In any case this results in a cycle containing $V(P)$ and e^* in H and hence an (a,b)-path of the same length as a longest (a,b)-path in H' (see Figure 2). $\square$

We can now prove our main result, Theorem 1.

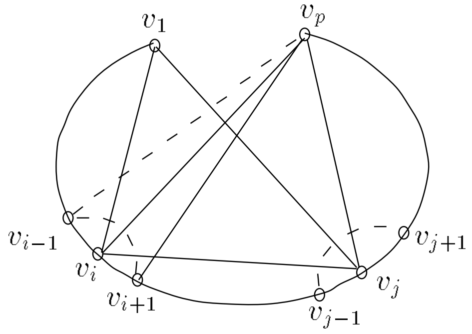

Figure 1: The edges and non-edges of G^*

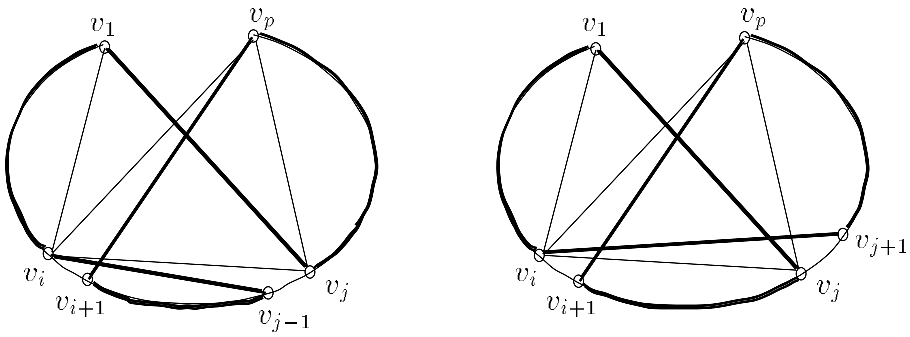

Figure 2: The two ways to obtain a cycle

Proof of Theorem 1.
Let G be a 9-connected claw-free graph and a, b be two vertices of G. Choose a vertex $x \neq a, b$ and fill all the $K_4 - e$'s whose centers are x and another vertex distinct from a, b. Then proceed to the next vertex until, finally, no induced $K_4 - e$ centered in two vertices distinct from a, b is left. By Lemma 1, in every step, the length of a longest (a, b)-path remains unchanged and all claws are centered at a or b. The resulting graph is by Theorem 2 the unique graph $G' = \mathrm{cl}_R(G, \{a, b\})$. Since this graph is obtained from G by adding edges it is 9-connected. Since G is 9-connected there are distinct neighbors u of a and v of b in G. Since $G' - a - b$ is 7-connected, claw-free and $K_4 - e$-free it is the line graph of a triangle-free graph, and by Zhan's result [10] Hamilton-connected. In particular, there is a hamiltonian (u, v)-path in $G' - a - b$, so there is a hamiltonian (a, b)-path in G' and therefore also in G. $\square$

Acknowledgement: I thank Wayne Goddard for proposing the stepwise approach to Ryjáček's closure and Hajo Broersma and Huib Trommel for pointing out a mistake in an earlier version.

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