SERIE A — MATHEMATIK

On the structure of dense triangle-free graphs

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Abstract

As a consequence of an early result of Pach we show that every maximal triangle-free graph is either homomorphic with a member of a specific infinite sequence of 3-colourable graphs or it contains the Petersen graph minus one vertex as a subgraph. From this result and further structural observations we derive that if a (not necessarily maximal) triangle-free graph of order n has minimum degree $\delta \geq n/3$ then the graph is either homomorphic with a member of the indicated family or contains the Petersen graph with one edge contracted. As a corollary we get a recent result due to Chen, Jin and Koh. Finally, we show that every triangle-free graph with $\delta > n/3$ is either homomorphic with C_5 or it contains the Möbius ladder as a subgraph. A major tool is the observation that every triangle-free graph with $\delta \geq n/3$ has a unique maximal triangle-free supergraph.

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1 Introduction and main results

We investigate properties of simple, finite and undirected graphs. For elementary concepts and notation the reader is referred to introductory literature, i.e. [16]. A graph is *maximal* triangle-free if it contains no triangle but joining any two non-adjacent vertices by an edge creates a triangle. Note that a triangle-free graph of order $n \geq 3$ is maximal if and only if it has diameter two. We investigate the structure of maximal triangle-free graphs and of triangle-free graphs with large minimum degree. It will turn out that the latter graphs are not too different from maximal triangle-free graphs since every such graph has a unique maximal triangle-free supergraph with which it shares many common properties. To state our results we need to introduce the notion of graph homomorphisms, as well as a sequence of specific triangle-free graphs which play an important role in the investigation of dense triangle-free graphs and which has been rediscovered several times over the years. As far as the author knows the sequence goes back to Andrasfai in 1964 [1].

Let $\Gamma_1 = K_2$ and $\Gamma_i = \overline{C_{3i-1}^{i-1}}$ for $i \geq 2$, i.e. Γ_i is the complement of the $i-1$st power of a cycle of length $3i-1$. As usual, the ith power of a graph G is the graph obtained by joining any pair of vertices at distance at most i in G. So $\Gamma_2 = C_5$ and Γ_3 is the Möbius ladder (with four spokes, see Figure 1). A homomorphism from a graph G to H is a mapping $\sigma : V(G) \to V(H)$ such that for every edge vw of G the edge $\sigma(v)\sigma(w)$ is present in H. Note that graph homomorphisms are a refinement of graph colouring, since a graph is k-colourable if and only if it is homomorphic with K_k. If H is triangle-free then G must be triangle-free as well, and if G is maximal triangle-free and H is triangle-free then $\sigma(u) = \sigma(v)$ implies $N_G(u) = N_G(v)$. Let H_{10} denote the Petersen graph, H_{10}/e the Petersen graph with one edge contracted (see Figure 2), $H_{10} - v$ the Petersen graph with one vertex deleted and H_{11} the Grötzsch graph (see Figure 3). Now we can state our main results.

Theorem 1 *Let G be a maximal triangle-free graph G. Then G is not homomorphic with any graph Γ_i if and only if G contains $H_{10} - v$ as a subgraph.*

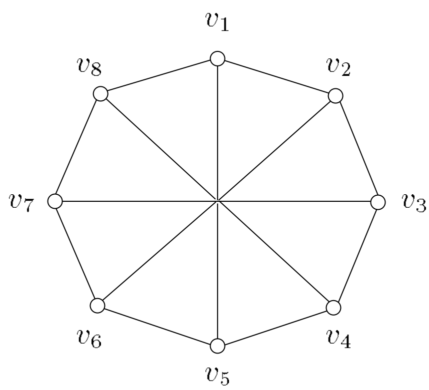

Figure 1: The Möbius ladder Γ_8.

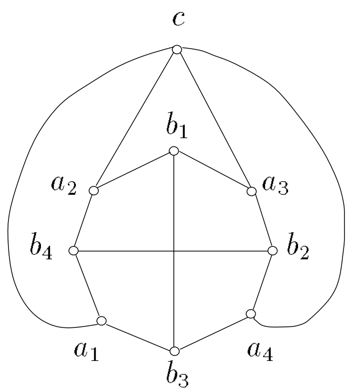

Figure 2: The edge-contracted Petersen graph H_{10}/e with certain labelling.

Theorem 2 *Let G be a triangle-free graph G of order n with minimum degree $\delta \geq n/3$. Then G is not homomorphic with any graph Γ_i if and only if G contains H_{10}/e as a subgraph.*

As a consequence, we can easily derive the main result of Chen, Jin and Koh [8] from Theorem 2, which only covers triangle-free graphs satisfying the slightly stronger requirement $\delta > n/3$. Note that the structural freedom seems to increase considerably when lowering the degree requirement from $\delta > n/3$ to $\delta \geq n/3$.

Corollary 1 (Chen, Jin and Koh [8]) *Let G be a triangle-free graph of order n with minimum degree $\delta > n/3$. Then G is not homomorphic with a graph in Γ_i if and only if G contains the Grötzsch graph H_{11} as a subgraph.*

This result distinguishes the 4-chromatic and the 3-colourable graphs, since the Grötzsch graph is 4-chromatic and the graphs in Γ_i are 3-colourable. Note that our approach is much simpler than the original proof by Chen, Jin and Koh which involves many lengthy and tedious calculations.

It can be seen by an easy counting argument that every triangle-free graph with $\delta > 2n/7$ is either bipartite (i.e. homomorphic with K_2) or contains C_5 as a subgraph. If $\delta(G) > n/3$ we can prove the next step.

Theorem 3 *Let G be a triangle-free graph with $\delta(G) > n/3$. Then G is not homomorphic with C_5 if and only if G contains Γ_3 as a subgraph.*

The result generalizes another result of Chen, Jin and Koh [8, Theorem 6], who essentially proved this statement for maximal triangle-free graphs. The degree bound is best possible, since the lexicographic product $H_{10}/e[\overline{K_r}]$ (see, e.g., [16, p. 370] for the definition) has minimum degree $n/3$, but is neither homomorphic with C_5 nor does it contain Γ_3.

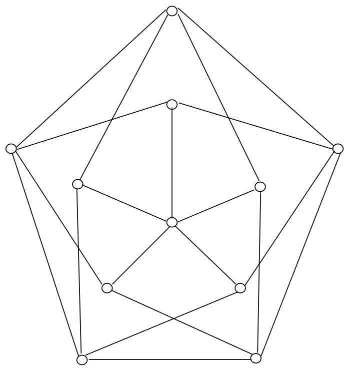

Figure 3: The Grötzsch graph H_{11}.

2 Background and history

There are many results showing that in triangle-free graphs with $\delta > n/3$ several graph parameters are strongly related among each other (see [4, Theorems 2.1, 2.2, 3.3 and Lemma 3.1] and [5]). Moreover, Goddard and Kleitman [10, Theorem 1] observed an interesting structural property of maximal triangle-free graphs satisfying the degree bound.

The investigation of dense triangle-free graphs relating degrees and homomorphisms has an interesting history of research progress obtained while dealing with false conjectures. The history started with a result of Andrasfai, Erdős and Sós [2], who proved—as a special case of a result for graphs of given clique number—that every triangle-free graph with $\delta > 2n/5$ is bipartite, i.e. homomorphic with K_2. Motivated by a construction of triangle-free graphs with $\delta > (1 - o(1))n/3$ with arbitrarily large chromatic number due to Erdős, Hajnal and Simonovits (see [9]) and unaware of a 4-chromatic triangle-free graph with $\delta > n/3$, Erdős and Simonovits [9] conjectured that every triangle-free graph with $\delta > (1 + o(1))n/3$ is 3-colourable. This conjecture was refuted by Häggkvist [11], who found a sequence of 4-chromatic graphs with $\delta = 10n/29$ homomorphic with the Grötzsch graph. Recently, with the help of the computer program Vega, Brandt and Pisanski [6] found an infinite sequence of homomorphically minimal subgraphs of triangle-free 4-chromatic graphs with $\delta > n/3$.

On the positive side, Häggkvist [11] proved that every triangle-free graph with $\delta > 3n/8$ is homomorphic with the Möbius ladder Γ_3 and conjectured that every triangle-free graph with $\delta > 10n/29$ is homomorphic with Γ_3. This, in turn, was refuted by Jin [12], who proved that every graph with $\delta > 10n/29$ is homomorphic with Γ_9 and therefore 3-colourable. On the other hand, Jin conjectured that there are triangle-free graphs with minimum degree exceeding $n/3$ and arbitrarily large chromatic number [13].

3 Methods and tools

There are two central features which make our proofs much simpler than those of Chen, Jin and Koh: **(i)** the existence of a unique maximal triangle-free supergraph for any triangle-free graph with $\delta \geq n/3$, which allows us to focus our attention to maximal triangle-free graphs, and the forbidden subgraph characterization in Theorem 1 of the triangle-free graphs for which **(ii)** every independent vertex set is contained in the neighbourhood of a vertex.

Let us first start with property (i), the existence of a unique maximal triangle-free supergraph. Closure concepts in graph theory proved to be useful tools mainly in connection with cycle lengths in graphs (see [3] and [15] for two particularly powerful closure concepts). Here we introduce a closure concept for triangle-free graphs with a slightly different flavour: we construct a supergraph $\mathrm{cl}_\Delta(G)$ by consecutively adding edges joining vertices of distance ≥ 3 until the resulting graph has diameter 2. In general, the result is far from being unique. But there is an easy characterization of those triangle-free graphs G, for which $\mathrm{cl}_\Delta(G)$ is unique and it is not difficult to show that this property is satisfied if $\delta(G) \geq n/3$.

Closely related to the closure is the *dist-2-graph*. For a given graph $G = (V, E)$, the edge set E_2 of the dist-2-graph $G_2 = (V, E_2)$ consists of the pairs of vertices which are at distance 2 in G. Note that a maximal triangle-free graph is the complement of its dist-2-graph. We will now characterize the triangle-free graphs which have a unique closure (in the labeled sense) via their dist-2-graphs.

Theorem 4 *Let G be a triangle-free graph. Then $\mathrm{cl}_\Delta(G)$ is uniquely determined if and only if $\overline{G_2}$ is triangle-free. In this case, $\mathrm{cl}_\Delta(G) = \overline{G_2}$.*

Proof. All edges appearing in some triangle-free supergraph of G are those joining vertices at distance 1 or at distance ≥ 3 in G, i.e. exactly those of $\overline{G_2}$. So if $\overline{G_2}$ is triangle-free we obtain that $\mathrm{cl}_\Delta(G) = \overline{G_2}$ is unique. Conversely, if $\overline{G_2}$ contains a triangle x, y, z, then at least two of its edges, say xy, xz, are not in G. So G has a maximal triangle-free supergraph G' containing xy and yz and another one $G'' \neq G'$ containing xz. $\square$

Next we show that for $\delta(G) \geq n/3$ the closure $\mathrm{cl}_\Delta(G)$ is indeed unique.

Corollary 2 *Let G be a triangle-free graph of order n. If $\delta(G) \geq n/3$ then $\mathrm{cl}_\Delta(G)$ is uniquely determined.*

Proof. Suppose that $\overline{G_2}$ has a triangle x, y, z with $xy, xz \notin E(G)$. Since no pair of these vertices has distance 2 in G, the neighbourhoods $N_G(x)$, $N_G(y)$, $N_G(z)$ are disjoint. Moreover, x is contained in none of the sets, since x has distance ≥ 3 to y and z, so $d_G(x) + d_G(y) + d_G(z) \leq n - 1$. $\square$

Note that $\mathrm{cl}_\Delta(G)$ shares many structural properties with G. I.e. it was shown in [5] that every independent set U with $|U| > n/3$ of G is an independent set of $\mathrm{cl}_\Delta(G)$ if $\delta(G) \geq n/3$.

Pach [14] investigated triangle-free graphs with the property (ii) and proved the following result, which is formulated here with regard to our needs (see also the simpler proof of this result due to Brouwer [7]).

Theorem 5 (Pach [14]) *Every independent set of a triangle-free graph G is contained in the neighbourhood of a vertex if and only if G is maximal triangle-free and homomorphic with a graph Γ_i.*

We will first give the simple proof of Theorem 1 based on Pach's result.

Proof of Theorem 1.

If every independent vertex set is contained in the neighbourhood of a vertex then, by Theorem 5, G is homomorphic with a graph Γ_i. Since in $H_{10} - v$ no two vertices can be identified by a homomorphism without creating a triangle and the subgraph induced by the degree 3 vertices of $H_{10} - v$ is an induced 6-cycle in any triangle-free supergraph, there must be an induced 6-cycle in G as well if $H_{10} - v$ is not homomorphic with G. So $H_{10} - v$ is not homomorphic with any graph Γ_i since it is easy to see that Γ_i does not contain an induced 6-cycle.

Now suppose that G contains an independent set U which is not contained in the neighbourhood of a vertex. Take a minimal independent subset $U' = \{u_1, \ldots, u_p\}$ of U which is not contained in the neighbourhood of a vertex. Since G has diameter 2 we have $p \geq 3$. By the minimality of U', for each $u_i \in U'$ there is a vertex w_i, which is adjacent to each vertex of U' except u_i. So the set $S = \{u_1, u_2, u_3, w_1, w_2, w_3\}$ induces a 6-cycle in G. Again by the diameter condition there must be distinct vertices v_1, v_2, v_3 such that v_i is adjacent to u_i and w_i. So the resulting subgraph is indeed $H_{10} - v$. $\square$

We cannot hope for a larger substructure since both the Petersen graph H_{10} and H_{10}/e are maximal triangle-free and are not homomorphic with any graph Γ_i. Note that by Theorem 1 we can reformulate Theorem 5 in the following way:

> Every independent set of a triangle-free graph G is contained in the neighbourhood of a vertex if and only if G is maximal triangle-free and does not contain $H_{10} - v$ as a subgraph.

4 Proofs

Now let us turn to the proof of Theorem 2. This is based on Corollary 2 and on a technical lemma which will be applied repeatedly. For a subgraph isomorphic to H_{10}/e we assume the labeling from Figure 2 to be given and we calculate the indices modulo 4.

Lemma 1 *Let G be a triangle-free graph of order n with $\delta(G) \geq n/3$. If the subgraph of $\mathrm{cl}_\Delta(G)$ induced by a subset $S \subseteq V(G)$ is H_{10}/e then G contains vertices y, z and indices i, j of different parity with $ya_i, yb_i, yb_{i+1}, za_j, zb_j, zb_{j+1} \in E(G)$, unless $\delta(G) = n/3$ and G contains all edges joining vertices of $S \setminus \{c\}$ in $\mathrm{cl}_\Delta(G)$.*

Proof. If all edges of $\mathrm{cl}_\Delta(G)$ joining vertices of $S \setminus \{c\}$ are present in G then there is nothing to prove, so assume that one of the edges, say e', is missing in G. Since the vertices a_i form an independent set we may assume that some vertex b_j is an endvertex of e'. First note that the subgraphs of $\mathrm{cl}_\Delta(G)$ induced by $S_1 = \{a_1, a_3, b_1, b_2, b_3, b_4\}$ and by $S_2 = \{a_2, a_4, b_1, b_2, b_3, b_4\}$ are both isomorphic with C_6 and that the other endvertex of e' has in G at most one neighbour in S_1 and in S_2. Since $\delta \geq n/3$ and not all vertices have exactly two neighbours in S_1 nor in S_2, there must be a vertex y having at least (and therefore by Corollary 2 exactly) three neighbours in S_1 and a vertex z having three neighbours in S_2, which have the required properties. □

Now we are prepared to prove Theorem 2 and Corollary 1. We will frequently switch between the graph G itself and the unique maximal triangle-free supergraph $\mathrm{cl}_\Delta(G)$.

Proof of Theorem 2.

If G contains H_{11} then G is not homomorphic with any graph Γ_i since H_{11} is 4-chromatic and Γ_i is 3-colourable. So suppose that G is not homomorphic with any graph Γ_i. Then, by the transitivity of the homomorphism relation, $\mathrm{cl}_\Delta(G)$ is also not homomorphic with a graph Γ_i. By Theorem 1, $\mathrm{cl}_\Delta(G)$ contains $H_{10} - v$ as a subgraph.

Consider the 6-cycle C of $H_{10} - v$ induced by the degree 3 vertices of $u_1, u_2, u_3, w_1, w_2, w_3$ with $u_i w_j \in E(C)$ iff $i \neq j$. By Corollary 2, the vertices u_i and w_i must have a common neighbour v_i in G, otherwise the edge $u_i w_i$ would be present in some closure. Since $\delta(G) \geq n/3$ and not all vertices v_i can have exactly 3 neighbours in S, there must be a vertex v having in G at least 4 neighbours in S. Since every independent set of 4 vertices of $H_{10} - v$ has the form $\{u_i, w_i, v_j, v_k\}$ with $\{i, j, k\} = \{1, 2, 3\}$, the set $S' = S \setminus \{v_i\} \cup \{v\}$ induces H_{10}/e in $\mathrm{cl}_\Delta(G)$ and induces in G all edges of H_{10}/e but possibly some edges of C.

For convenience, relabel the vertices of S' according to the labeling of H_{10}/e in Figure 2. We claim that G contains an induced 6-cycle which is an induced 6-cycle in $\mathrm{cl}_\Delta(G)$ as well. If all the edges of H_{10}/e are present in G, this is certainly the case. Otherwise, by Lemma 1, there are vertices y, z adjacent to a_i, b_i, b_{i+1} and a_j, b_j, b_{j+1} in G, where i and j are of different parity. So the subgraph of G induced by $S' \cup \{y, z\}$ is H_{11} with some edges of an induced 6-cycle of H_{11} missing. Since for every induced 6-cycle of H_{11} there is another edge disjoint induced 6-cycle, we obtain that G and $\mathrm{cl}_\Delta(G)$ contain a common induced 6-cycle C'. By repeating the argument of the second paragraph of this proof with C' in place of C, we obtain that G contains H_{10}/e. □

Proof of Corollary 1.

If G is not homomorphic with a graph Γ_i then by Theorem 2 it must contain H_{10}/e

as a subgraph. By Lemma 1 applied to the subgraph H_{10}/e, the graph G contains H_{11}, as claimed. $\square$

Note that in Corollary 1 we have proven the main result of [8] as stated in the abstract of [8]. In the paper they prove the slightly stronger but somewhat more technical result [8, Theorem 3], saying that any graph with $\delta > n/3$ which is triangle-free and does not contain H_{11}, and which is edge-maximal subject to these properties, is either homomorphic with Γ_{i-1} or contains Γ_i as a subgraph for every $i \geq 2$. This can be easily derived from our results. By Corollary 1 such a graph is maximal triangle-free and homomorphic with some graph Γ_i. Choose i to be the smallest integer such that G is homomorphic with Γ_i. Since Γ_i is a subgraph of Γ_j whenever $j \geq i$, we get by the transitivity of homomorphisms that G is also homomorphic with Γ_j. If the homomorphism $\sigma : V(G) \to V(\Gamma_i)$ is surjective then G contains Γ_i since both graphs are maximal triangle-free. Otherwise, G is homomorphic with $\Gamma_i - v$, and since $\Gamma_i - v$ is homomorphic with Γ_{i-1} we get that G is homomorphic with Γ_{i-1}, a contradiction to the choice of i.

Finally, we will give the simple proof of Theorem 3.

Proof of Theorem 3.
Let G contain H_{11} as a subgraph then G contains Γ_3, since H_{11} contains Γ_3. Otherwise, by Corollary 1 the graph G is homomorphic with Γ_i for some i. Let j be the smallest integer, such that G is homomorphic with Γ_j. If $j \leq 2$ then we are done so suppose $j \geq 3$. As above we may assume that the homomorphism $\sigma : V(G) \to V(\Gamma_j)$ is surjective, otherwise G is homomorphic with Γ_{j-1}. By adding all edges to G, joining vertices v, w with $\sigma(v)\sigma(w) \in E(\Gamma_j)$ we get a maximal triangle-free supergraph $\mathrm{cl}_\Delta(G)$ which is by Corollary 2 unique. The graph $\mathrm{cl}_\Delta(G)$ contains Γ_j as a subgraph and therefore Γ_3 as well.

Let G' be a subgraph of G which induces Γ_3 in $\mathrm{cl}_\Delta(G)$ and has, subject to this property, as many edges as possible. We will show that G' is isomorphic to Γ_3. Note that there are two edge orbits in Γ_3, the outer cycle and the diagonals in Figure 1. Suppose that an edge of the outer cycle, v_1v_2 say, is missing in G'. By the uniqueness of $\mathrm{cl}_\Delta(G)$ there must be a common neighbour x of v_1 and v_3 in G, otherwise we could add the edge v_1v_3. It is easy to check that $x \not\subseteq V(G')$. So the subgraph of $\mathrm{cl}_\Delta(G)$ induced by $V(G') \setminus \{v_2\} \cup \{x\}$ contains Γ_3 with the diagonal xv_6 deleted. Now consider the 6 vertices $S = \{v_1, x, v_3, v_5, v_6, v_7\}$. Since $\delta(G) > n/3$ there must be a vertex $y \in V(G)$ (possibly x or v_6) having 3 neighbours in G. But then, by replacing x or v_6 by y we get a subgraph G'' which induces Γ_3 in $\mathrm{cl}_\Delta(G)$ and has more edges than G', a contradiction to the choice of G'. Finally, if a diagonal, say v_2v_6, is missing, then some vertex y has 3 neighbours in $S' = \{v_1, v_2, v_3, v_5, v_6, v_7\}$ and again, replacing v_2 or v_6 by y we get a subgraph with more edges than G' inducing Γ_3 in $\mathrm{cl}_\Delta(G)$. So, indeed, G contains Γ_3 as a subgraph. $\square$

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