Estimation of variance components with high breakdown point and high efficiency

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Abstract. For the 1-way random effect model three noniterative robust variance component estimators which were recently proposed by Uhlig (1995) are investigated. For these estimators block breakdown points, measurement breakdown points and efficiency are derived. It is shown that the block breakdown points approximate 50% and that the measurement breakdown points are not smaller than 25%. By deriving the asymptotic normal distribution of these estimators it turns out that two of the new estimators have an asymptotic efficiency greater than 82% and under certain circumstances up to 95%. A simulation study shows that the high efficiency also appears for finite samples. The use of the estimators for interlaboratory tests is demonstrated in a practical example.

1. INTRODUCTION

Consider the balanced 1-way random effect model

(1.1) $Y_{ni} = \mu + X_n + Z_{ni},$

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where $n = 1, \ldots, N$ denotes the block (factor level) and $i = 1, \ldots, I$ denotes the single measurement (replicate). The random variables $X_1, \ldots, X_N, Z_{11}, \ldots, Z_{NI}$ are stochastically independent. Moreover, in an ideal model it is assumed that they are normally distributed with expectation zero and variances $Var[X_n] = \sigma_1^2$ and $Var[Z_{ni}] = \sigma_0^2$. Then the variance of Y_{ni} is $\sigma_Y^2 := Var(Y_{ni}) = \sigma_0^2 + \sigma_1^2$.

Let $Y_n = (Y_{n1}, ..., Y_{nI})^\top$ denote the vector of observations in the block n and y_n, y_{ni}, x_n and z_{ni} the realizations of Y_n, Y_{ni}, X_n and Z_{ni}, respectively. Under the normality asymptions the unknown variance components σ_0^2 and σ_Y^2 can be estimated by the classical ANOVA (maximum likelihood) estimators

$$(1.2) \quad \widehat{\sigma}_0^2(y_1, \ldots, y_N) = \frac{1}{N} \sum_{n=1}^{N} \widehat{Var}(y_{n1}, \ldots, y_{nI})$$

and

$$(1.3) \quad \widehat{\sigma}_Y^2(y_1, \ldots, y_N) = \frac{I-1}{I} \widehat{\sigma}_0^2(y_1, \ldots, y_N) + \widehat{Var}\left(\frac{1}{I} \sum_{i=1}^{I} y_{1i}, \ldots, \frac{1}{I} \sum_{i=1}^{I} y_{Ni} \right),$$

where $\widehat{Var}$ provides the empirical variance.

For example ISO-5725-2 (1992), the international protocol for the validation of e.g. chemical analytical methods, demands the use of $\widehat{\sigma}_0^2$ and $\widehat{\sigma}_Y^2$. In particular, $\widehat{\sigma}_0^2$ and $\widehat{\sigma}_Y^2$ are used for interlaboratory comparison tests: In order to assess the performance of individual analytical laboratories, several laboratories $n = 1, \ldots, N$ analyse samples of a homogenous test material and obtain measurement values Y_{ni}. However, the use of the classical ANOVA estimators implies that all laboratories have the same performance level. This is not consistent with the idea of assessing performance levels of the individual laboratories. If there are just a few 'outlier' laboratories, the data of these laboratories can be eliminated by outlier tests as described in ISO-5725-2. But in interlaboratory comparison tests employed for the certification and licencing of laboratories, there is quite often a large number (often more than 25%) of laboratories which do not meet the quality requirements (see Uhlig and Lischer (1997)). This demands for outlier robust variance component estimators and in particular for estimators with high breakdown point.

There are a lot of papers proposing robust variance component estimators by generalizing maximum likelihood estimators to M-estimators with bounded influence function (see Rocke 1983, 1991, Fellner 1986, Huggins 1993a,b, Richardson and Welsh 1995, Bednarski and Zontek 1996 and Richardson 1997). But considerations concerning the breakdown points of these estimators are not made. Recently Uhlig (1995, 1997a,b) and Lischer (1996) proposed different variance component estimators based on the Q estimator of Rousseeuw and Croux (1992, 1993), which is a high breakdown point scale estimator.

In opposite to the robust variance components estimators based on M-estimators these estimators can be calculated without iteration and they have a simple structure. In particular the estimators of Uhlig (1995, 1997a,b) are based without exception on differences of the single measurements. As the ANOVA estimators split the overall variance into the variance within blocks and variance between blocks, these new estimators partition the set of all differences into differences within blocks and differences between blocks. Therefore the new estimators are easy to understand for practitioners. Per extensive simulation studies Uhlig found that the new estimators have good efficiency properties and provide high outlier robustness. Meanwhile the new estimators are also recommended in a draft of the LAWA (Cooperation of the German "Bundesländer" for water analysis), in a draft of the VDLUFA (the German research agencies for agriculture) for licencing of private chemical laboratories and in a booklet of the German Federal Agency of Environment (Uhlig 1997c).

Up to now only few theoretical properties of the new estimators are known. Namely, only in Uhlig (1997a,b) consistency and some breakdown point properties of these estimators are presented. Thereby Uhlig (1997a,b) pointed out that the notion of breakdown point has to be redefined because two types of outlier can appear in a 1-way random effect model: outlying blocks and single outliers within blocks. While it is rather easy to see that the breakdown point against outlying blocks (also called block breakdown point) is approximately 50% it is difficult to calculate the breakdown point against single outliers. In Section 3 of this paper we calculate the breakdown point against single outliers (also called measurement breakdown point) in full detail and we show that this breakdown point lies between 25% and 50%.

In Section 4 we also study the efficiency properties of the new estimators. For calculating the asymptotic efficiencies we derive the asymptotic normality of the estimators by extending a result of Serfling (1984) concerning generalized L-statistics. It is shown that the efficiency of two of the new estimators is greater than 82%. This high efficiency is supported by the simulation study presented in Section 5.

Additionally, in Section 6 an example concerning an interlaboratory comparison test based on the analysis of arsenic in soil is presented. The example in particular shows that the rate of seemingly outlying laboratories (blocks) exceeds 30% and that analysing the data along ISO-5725-2 (1992) provides misleading results.

All proofs are given in the appendix.

2. THE ESTIMATORS

Let Φ denote the distribution function of the standard normal distribution. For estimating σ_0 the following estimators are proposed (see Uhlig 1995, 1997a,b):

$$S_{N,1}(y_1, \ldots, y_N) := \frac{1}{\sqrt{2}\Phi^{-1}(3/4)}\ H_{N,1}^{-1}(0.5) = 1.048\ H_{N,1}^{-1}(0.5),$$

$$S_{N,2}(y_1, \ldots, y_N) := \frac{1}{2\Phi^{-1}(5/8)}\ H_{N,2}^{-1}(0.25) = 1.569\ H_{N,2}^{-1}(0.25),$$

where

$$H_{N,1}(x) := \frac{1}{N} \sum_{n=1}^{N} \frac{2}{I(I-1)} \sum_{1 \leq i < j \leq I} 1\{|y_{ni} - y_{nj}| \leq x\}$$

$$H_{N,2}(x) := \frac{1}{\binom{N}{2}} \sum_{1 \leq n < m \leq N} \frac{2}{I^2(I-1)^2} \sum_{1 \leq i < j \leq I} \sum_{1 \leq k < l \leq I} 1\{|y_{ni} - y_{nj} - y_{mk} + y_{ml}| \leq x\}$$
$$+ 1\{|y_{ni} - y_{nj} + y_{mk} - y_{ml}| \leq x\}.$$

$H_{N,1}(x)$ can be interpreted as the distribution function of the absolute intra-block differences and $S_{N,1}$ is simply a Fisher-consistent version of the median of all these differences. $H_{N,2}(x)$ equals the distribution function of all absolute inter-block differences of the intra-block differences, which are also called second order differences. In that case the estimator $S_{N,2}$ corresponds to the lower quartile of these second order differences. We mainly recommend the estimator $S_{N,2}$ while the estimators $S_{N,1}$ is useful for understanding the construction (see also below).

For $I = 2$ the estimator $S_{N,2}$ equals the lower quartile of all absolute differences $|y_{n1} - y_{n2} - y_{m1} + y_{m2}|$ and $|y_{n1} - y_{n2} + y_{m1} - y_{m2}|$. This is similar but not equal to an estimator proposed by Lischer (1996), which is the Q estimator of $y_{11} - y_{12}, y_{21} - y_{22}, \ldots, y_{N1} - y_{N2}$, i.e. the $k/\binom{N}{2}$ quantile of $y_{11} - y_{12}, y_{21} - y_{22}, \ldots, y_{N1} - y_{N2}$, where $k = \binom{h}{2} \approx \binom{N}{2}/4$ and $h = \lfloor N/2 \rfloor + 1$ (see also Rousseeuw and Croux 1992, 1993). While $S_{N,1}$ and $S_{N,2}$ are invariant with respect to permutations of y_{n1} and y_{n2} this it not the case for the estimator of Lischer.

For estimating σ_Y the following estimator is proposed (see Uhlig 1995, 1997a,b):

$$S_{N,3}(y_1, \ldots, y_N) := \frac{1}{\sqrt{2}\Phi^{-1}(5/8)}\ H_{N,3}^{-1}(0.25) = 2.219\ H_{N,3}^{-1}(0.25),$$

where

$$H_{N,3}(x) := \frac{1}{\binom{N}{2}} \sum_{1 \leq n < m \leq N} \frac{1}{I^2} \sum_{i=1}^{I} \sum_{j=1}^{I} 1\{|y_{ni} - y_{mj}| \leq x\}.$$

Like $H_{N,1}$ and $H_{N,2}$, the function $H_{N,3}(x)$ can be interpreted as the distribution function of the absolute inter-block differences.

For the case of two replicates, i.e. $I = 2$, Lischer (1996) proposed a different estimator for σ_Y by basing the estimator on the Q estimator of $(y_{11} + y_{12})/2, \dots, (y_{N1} + y_{N2})/2$. Below we will see that this estimator is less efficient and has a smaller breakdown point.

Regarding $S_{N,1}$ together with $S_{N,3}$ we see that the set of all absolute differences $|y_{ni} - y_{mj}|$, $n, m = 1, \dots, N$, $l, j = 1, \dots, I$ is partitioned into intra-block differences and inter-block differences providing $S_{N,1}$ and $S_{N,3}$ as the ANOVA estimators do with the overall variance. Concerning the estimator $S_{N,2}$, which is based on second order intra-class differences $|y_{ni} - y_{nj} - y_{mk} + y_{ml}|$, we could regard also a corresponding estimator $S_{N,4}$ based on second order inter-class differences $|y_{n_1 i} - y_{n_2 j} - y_{m_1 k} + y_{m_2 l}|$. But such estimator is difficult to calculate, and it is also not necessary since already $S_{N,3}$ has a high breakdown point and a high efficiency, which is not the case for $S_{N,1}$. For that see Section 3 and Section 4.

Note, that the finite sample efficiency of $S_{N,2}$ and $S_{N,3}$ can be improved by using instead of the 0.25 quantile the quantile which Rousseeuw and Croux (1992, 1993) used for the Q estimator. But asymptotically this makes no difference.

The estimators $S_{N,1}$, $S_{N,2}$ and $S_{N,3}$ can be defined also for models which are not balanced, and it is also possible to extend them to hierarchical models. See Uhlig (1997a,b).

3. BREAKDOWN POINTS

The breakdown point of an estimator is the smallest proportion of observations replaced by outliers that can cause a 'breakdown' of the estimator. Several breakdown point definitions are available. For (location) estimators see Donoho and Huber (1983), Rousseeuw and Leroy (1987), He et al. (1990), Ellis and Morgenthaler (1992), Müller (1997), and for scale estimators in the univariate case see Rousseeuw and Croux (1992, 1993). However, these concepts cannot be applied directly for the 1-way random effect model, because in the latter two sources of error, blocks and replicates, have to be taken into account (see Uhlig 1997a,b). Associated with outlying blocks is the following definition. It is a replacement version of the breakdown point based on the observation vectors $y_n = (y_{n1}, \dots, y_{nI})^\top$, $n = 1, \dots, N$:

Definition 1. *The block breakdown point of the scale estimator* $S : I\!R^{I\times N} \to I\!R^+$ *for the error variance* σ_0^2 *or for the total variance* σ_Y^2 *at* $(y_1,\dots,y_N) \in I\!R^{I\times N}$ *is defined by*

$$\begin{aligned} e_N^b(S,(y_1,\dots,y_N)) = \frac{1}{N}\min\{M;\ & \\ & \sup\left\{S(\overline{y}_1,\dots,\overline{y}_N);\ (\overline{y}_1,\dots,\overline{y}_N) \in L_{N,M}^b(y_1,\dots,y_N)\right\} = \infty \ \textit{or} \\ & \inf\left\{S(\overline{y}_1,\dots,\overline{y}_N);\ (\overline{y}_1,\dots,\overline{y}_N) \in L_{N,M}^b(y_1,\dots,y_N)\right\} = 0\right\} \end{aligned}$$

where $L_{N,M}^b(y_1,\dots,y_N)$ *denotes the set of all samples that are obtained by replacing* M *data vectors* $y_{k_1},\dots,y_{k_M}$ *by arbitrary vectors of same size, i.e.*

$$\begin{aligned} L_{N,M}^b(y_1,\dots,y_N) = \Big\{(\overline{y}_1,\dots,\overline{y}_N) \in I\!R^{I\times N};& \\ & \#\{n;\ n=1,\dots,N,\ \overline{y}_n \neq y_n\} \le M\Big\} \end{aligned}$$

The block breakdown point measures the smallest proportion of contaminated blocks that can cause the estimated value tending to ∞ or to 0.

Associated with the observations itselves is the measurement breakdown point:

Definition 2. *The measurement breakdown point of the scale estimator* $S : I\!R^{I\times N} \to I\!R^+$ *for the error variance* σ_0^2 *or for the total variance* σ_Y^2 *at* $(y_1,\dots,y_N) \in I\!R^{I\times N}$ *is defined by*

$$\begin{aligned} e_N^m(S,(y_1,\dots,y_N)) = \frac{1}{I\cdot N}\min\{M;\ & \\ & \sup\left\{S(\overline{y}_1,\dots,\overline{y}_N);\ (\overline{y}_1,\dots,\overline{y}_N) \in L_{N,M}^m(y_1,\dots,y_N)\right\} = \infty \ \textit{or} \\ & \inf\left\{S(\overline{y}_1,\dots,\overline{y}_N);\ (\overline{y}_1,\dots,\overline{y}_N) \in L_{N,M}^m(y_1,\dots,y_N)\right\} = 0\right\} \end{aligned}$$

where $L_{N,M}^m(y_1,\dots,y_N)$ *denotes the set of all samples that are obtained by replacing* M *observations* y_{ni} *by arbitrary variables, i.e.*

$$\begin{aligned} L_{N,M}^m(y_1,\dots,y_N) = \Big\{(\overline{y}_1,\dots,\overline{y}_N) \in I\!R^{I\times N};& \\ & \#\{(n,i);\ n=1,\dots,N,\ i=1,\dots,I_n,\ \overline{y}_{ni} \neq y_{ni}\} \le M\Big\} \end{aligned}$$

The measurement breakdown point measures the smallest proportion of contaminated observations that can cause the estimated value tending to ∞ or to 0.

The following inequality is obvious:

Lemma 1. *For every estimator* S

$$e_N^b(S,(y_1,\dots,y_N)) \ge e_N^m(S,(y_1,\dots,y_N)).$$

Theorem 1 (Breakdown points of $S_{N,1}$). *At any sample* $(y_1, \dots, y_N)$ *in which all observations* $\{y_{ni}\}$ *are pairwise different, the block breakdown point of* $S_{N,1}$ *satisfies*

$$\lim_{N\to\infty} e_N^b(S_{N,1}, (y_1, \dots, y_N)) = \frac{1}{2},$$

and the measurement breakdown point of $S_{N,1}$ *satisfies*

$$\lim_{N\to\infty} e_N^m(S_{N,1}, (y_1, \dots, y_N)) \geq \frac{1}{4}.$$

Theorem 2 (Breakdown points of $S_{N,2}$). *At any sample* $(y_1, \dots, y_N)$ *in which all differences* $|y_{ni} - y_{nj}|$ *are pairwise different, the block breakdown point of* $S_{N,2}$ *satisfies*

$$\lim_{N\to\infty} e_N^b(S_{N,2}, (y_1, \dots, y_N)) = \frac{1}{2},$$

and the measurement breakdown point of $S_{N,2}$ *satisfies*

$$\lim_{N\to\infty} e_N^m(S_{N,2}, (y_1, \dots, y_N)) \geq \frac{1}{4}.$$

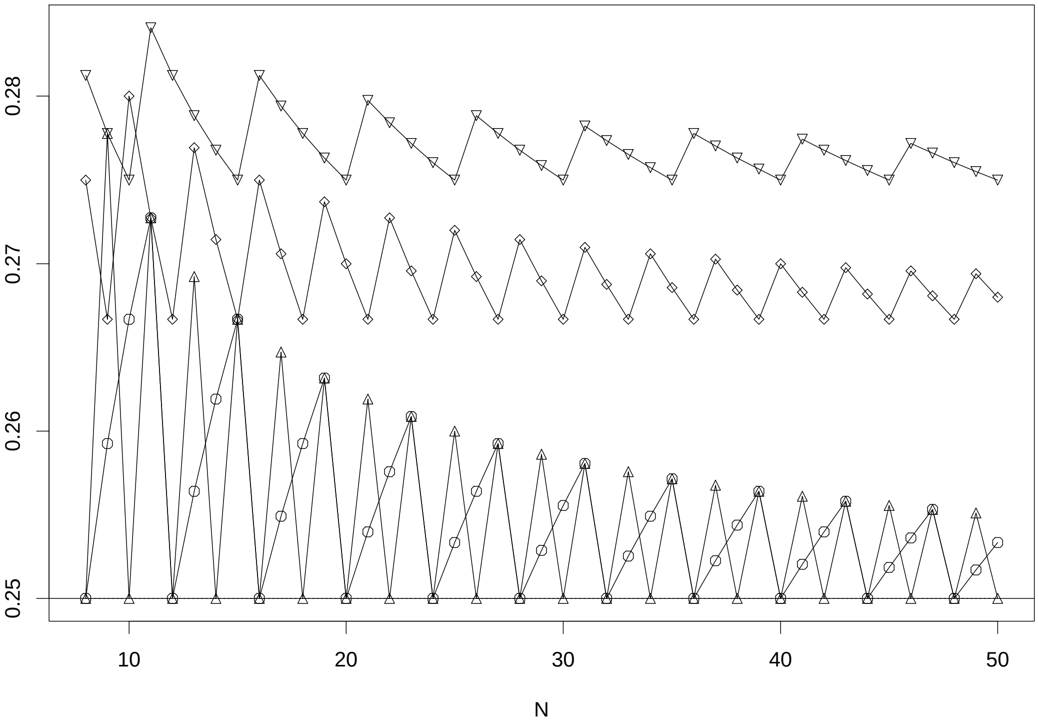

FIGURE 1. Measurement breakdown points of $S_{N,2}$ for $I = 2$ ($\triangle$), $I = 3$ ($\bigcirc$), $I = 5$ ($\diamond$) and $I = 8$ ($\bigtriangledown$)

The proofs of Theorem 1 and Theorem 2 show how to calculate the breakdown points of $S_{N,1}$ and $S_{N,2}$ for finite sample sizes N. For $I = 2, 3, 5, 8$ and $N = 8, \ldots, 50$ the measurement breakdown points of $S_{N,2}$ are given in Figure 1. A similar figure with little improvements holds for $S_{N,1}$. For $I = 2$ the lower bound of $\frac{1}{4}$ is sharp for $S_{N,1}$ as well as for $S_{N,2}$ while for $I = 3$ it is only sharp for $S_{N,2}$. For $I > 4$ an improved lower bound for both estimators is $\frac{1}{I}\lfloor I - \frac{1}{2} - \frac{1}{2}\sqrt{2I^2 - 2I + 1}\rfloor$. Note also, that for $I = 2$ the estimator of Lischer (1996) for σ_0 has the same breakdown points as $S_{N,2}$.

Theorem 3 (Breakdown points of $S_{N,3}$). *At any sample* $(y_1, \ldots, y_N)$ *in which all observations* $\{y_{ni}\}$ *are pairwise different, the block breakdown point of* $S_{N,3}$ *satisfies*

$$\lim_{N\to\infty} e_N^b(S_{N,3}, (y_1, \ldots, y_N)) = \frac{1}{2},$$

and the measurement breakdown point of $S_{N,3}$ *satisfies*

$$\lim_{N\to\infty} e_N^m(S_{N,3}, (y_1, \ldots, y_N)) = \frac{1}{2}.$$

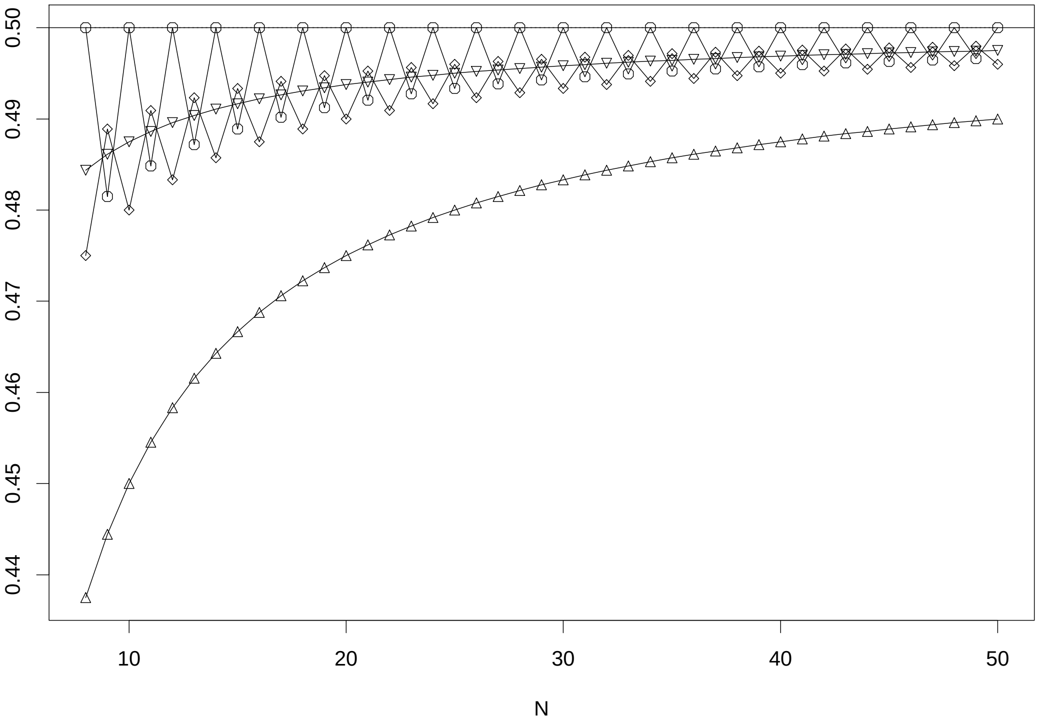

FIGURE 2. Measurement breakdown points of $S_{N,3}$ for $I = 2$ (△), $I = 3$ (◯), $I = 5$ (◇) and $I = 8$ (▽)

While the estimator of Lischer (1996) for σ_Y, based on $(y_{11}+y_{12})/2, \ldots, (y_{N1}+y_{N2})/2$, has the same block breakdown point as $S_{N,3}$ its measurement breakdown is worse, namely $\frac{1}{4}$ instead of $\frac{1}{2}$. As for $S_{N,1}$ and $S_{N,2}$ the breakdown points of $S_{N,3}$ for finite N can be calculated from the derivations in the proof of Theorem 3. Figure 2 shows that in particular the measurement breakdown points for $N \geq 8$ are not smaller than 0.4 and that they converge quickly to the maximum breakdown point of $\frac{1}{2}$.

4. ASYMPTOTIC EFFICIENCY

To derive the asymptotic normality and thus the asymptotic efficiency we assume that the number N of blocks is going to infinity. Another possibility would be the assumption that the number I of repetitions is going to infinity. But this would not meet the practical requirements, in particular for interlaboratory tests, where N is large (typically between 15 and 150) and I is small (typically we have $2 \leq I \leq 6$). Let ϕ denote the density of the standard normal distribution $\mathcal{N}(0,1)$.

Theorem 4. *Assume that $X_1, \ldots, X_N, Z_{11}, \ldots, Z_{NI}$ of the model (1.1) have a normal distribution. Then $\sqrt{N}(S_{N,1} - \sigma_0)$ has an asymptotic normal distribution with mean* 0 *and variance*

$$\frac{\sigma_0^2}{(2\Phi^{-1}(3/4)\,\phi(\Phi^{-1}(3/4)))^2}\left(\frac{1}{2\,I\,(I-1)} + \frac{4(I-2)}{I(I-1)}\left(v_1 - \frac{1}{4}\right)\right),$$

where

$$v_1 := \int \left[\Phi\left(z + \sqrt{2}\Phi^{-1}(3/4)\right) - \Phi\left(z - \sqrt{2}\Phi^{-1}(3/4)\right)\right]^2 \phi(z)dz \approx 0.276.$$

Theorem 5. *Assume that $X_1, \ldots, X_N, Z_{11}, \ldots, Z_{NI}$ of the model (1.1) have a normal distribution. Then $\sqrt{N}(S_{N,2} - \sigma_0)$ has an asymptotic normal distribution with mean* 0 *and variance*

$$\frac{\sigma_0^2}{(\Phi^{-1}(5/8)\,\phi(\Phi^{-1}(5/8)))^2}\left(\frac{2}{I(I-1)}\left(v_2 - \frac{1}{16}\right) + \frac{4(I-2)}{I(I-1)}\left(v_3 - \frac{1}{16}\right)\right),$$

where

$$v_2 := \int \left[\Phi\left(z + \sqrt{2}\Phi^{-1}(5/8)\right) - \Phi\left(z - \sqrt{2}\Phi^{-1}(5/8)\right)\right]^2 \phi(z)dz \approx 0.071,$$

and

$$v_3 := \int \Big[\int \Big[\Phi\left(z_2/\sqrt{2} - z_1/\sqrt{2} + \sqrt{2}\Phi^{-1}(5/8)\right) - \Phi\left(z_2/\sqrt{2} - z_1/\sqrt{2} - \sqrt{2}\Phi^{-1}(5/8)\right)\Big]\phi(z_1)dz_1\Big]^2 \phi(z_2)dz_2 \approx 0.064.$$

Comparing the asymptotic variances of the robust estimators with the asymptotic variances of the ANOVA estimates $\widehat{\sigma}_0$ given by (1.2) and $\widehat{\sigma}_Y$ given by (1.3) we obtain the efficiencies. Figure 3 presents the asymptotic efficiencies for $S_{N,1}$ and $S_{N,2}$. For $I \to \infty$ it can be shown that the efficiency of $S_{N,1}$ exceeds 86% and that the efficiency of $S_{N,2}$ approximates 96%.

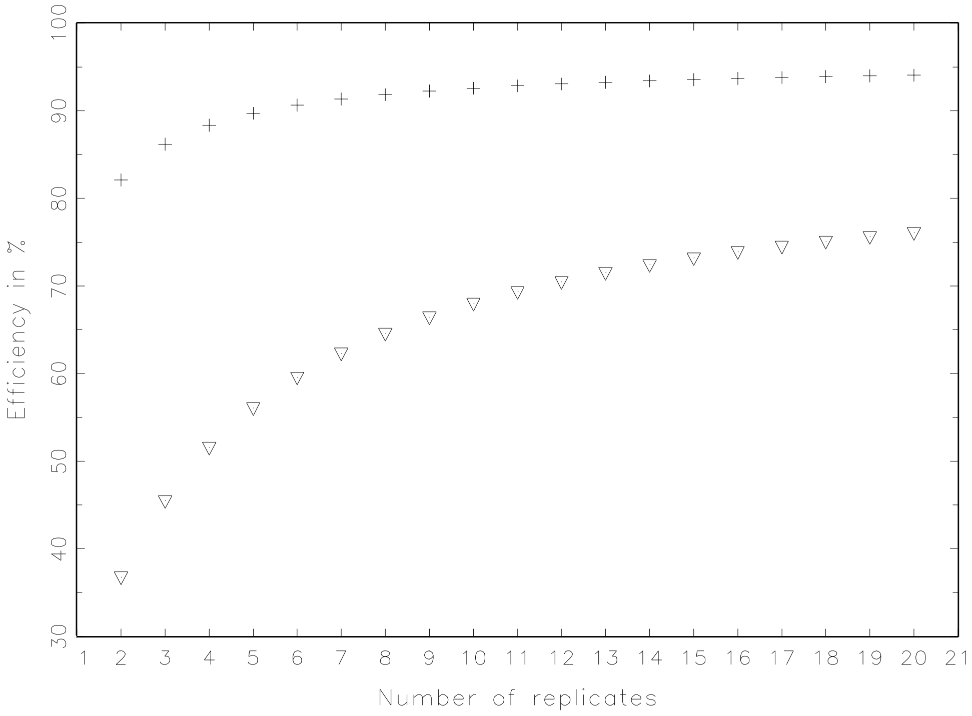

FIGURE 3. Asymptotic efficiencies of $S_{N,1}$ ($\bigtriangledown$) and $S_{N,2}$ ($+$)

Theorem 6. *Assume that* $X_1, \dots, X_N, Z_{11}, \dots, Z_{NI}$ *of the model (1.1) have a normal distribution. Then* $\sqrt{N}(S_{N,3} - \sqrt{\sigma_1^2 + \sigma_0^2})$ *has an asymptotic normal distribution with mean* 0 *and variance*

$$\frac{\sigma_1^2 + \sigma_0^2}{\left(\Phi^{-1}(5/8)\,\phi(\Phi^{-1}(5/8))\right)^2}\; Var\left(\frac{1}{I}\sum_i^I \left[\Phi\left(\frac{Y_{1i}}{\sqrt{\sigma_1^2+\sigma_0^2}} + \sqrt{2}\Phi^{-1}(5/8)\right) - \Phi\left(\frac{Y_{1i}}{\sqrt{\sigma_1^2+\sigma_0^2}} - \sqrt{2}\Phi^{-1}(5/8)\right)\right]\right).$$

Unfortunately the calculation of the asymptotic variance of $S_{N,3}$ requires more computational effort. For calculating the asymptotic efficiencies in Figure 4 we derived the variance expression in Theorem 6 by simulation over 200 000 samples. Note, that in this case the efficiencies depend on the ratio of σ_1 and σ_0 so that the efficiencies are only given for $I = 2$, $I = 4$ and $I = 10$.

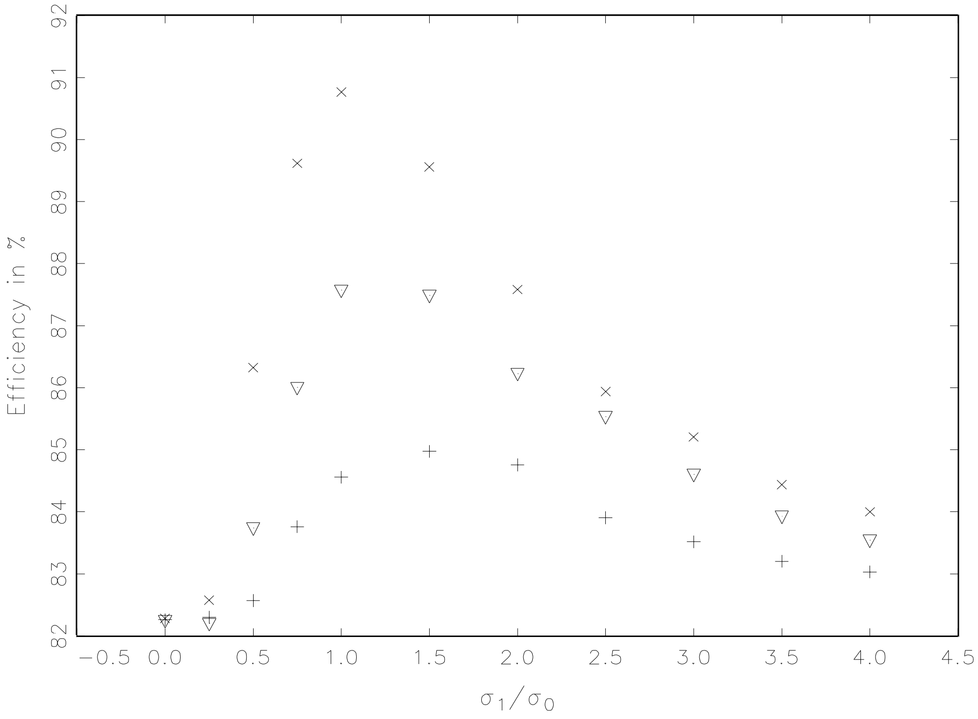

FIGURE 4. Asymptotic efficiencies of $S_{N,3}$ for $I = 2$ (+), $I = 4$ ($\bigtriangledown$) and $I = 10$ ($\times$)

5. SIMULATION STUDY AND DISCUSSION

In order to get insight into the finite sample properties of the robust estimators, a simulation study was performed with 10000 runs per parameter setting. The following tables list the results for the balanced model with $I = 2$ replicates. The calculation of the efficiency is based on the standardized variance

$$\text{Std.Var.}(S_N) = N \cdot Var(S_N) / \left(Mean(S_N)\right)^2$$

where $Var(S_N)$ and $Mean(S_N)$ denote the values obtained from the simulation study. It was argued by Bickel and Lehmann (1976) that this standardization is needed to obtain a natural measure of accuracy for scale estimators. Hence, the efficiency is

$$\frac{\text{Std.Var.}(\widehat{\sigma}_0)}{\text{Std.Var.}(S_N)} \text{ and } \frac{\text{Std.Var.}(\widehat{\sigma}_Y)}{\text{Std.Var.}(S_N)}$$

for estimating σ_0 and σ_Y, respectively, where $\widehat{\sigma}_0$ and $\widehat{\sigma}_Y$ are the ANOVA estimates given by (1.2) and (1.3).

For estimating σ_0 the standardized variance is independent on σ_0 and σ_1 and the mean Mean$(\cdot)$ depends only on σ_0. For the following simulation results $\sigma_0 = 1$ was used.

Simulation results of $S_{N,1}$

	$N=8$	$N=12$	$N=16$	$N=20$	$N=30$	$N=50$	$N=150$	$N \to \infty$
Std.Var.:	1.041	1.121	1.168	1.222	1.273	1.342	1.351	
Mean:	1.044	1.026	1.020	1.016	1.013	1.007	1.003	
Eff.:	49.6%	45.2%	43.5%	41.5%	39.1%	38.4%	37.1%	36.7%

Simulation results of $S_{N,2}$

	$N=8$	$N=12$	$N=16$	$N=20$	$N=30$	$N=50$	$N=150$	$N \to \infty$
Std.Var.:	0.721	0.703	0.679	0.670	0.662	0.647	0.620	
Mean:	1.310	1.202	1.145	1.118	1.079	1.045	1.015	
Eff.:	71.6%	74.4%	74.8%	75.7%	78.0%	79.6%	80.9%	82.2%

Simulation results of $S_{N,Lischer}$

	$N=8$	$N=12$	$N=16$	$N=20$	$N=30$	$N=50$	$N=150$	$N \to \infty$
Std.Var.:	0.943	0.882	0.796	0.787	0.734	0.706	0.641	
Mean:	1.496	1.32	1.235	1.191	1.127	1.074	1.025	
Eff.:	54.7%	59.5%	63.1%	64.4%	67.8%	72.9%	78.2%	82.2%

It is interesting that the estimator $S_{N,Lischer}$ is less efficient than $S_{N,2}$, although its asymptotic efficiency is the same which can be seen from the proof of Theorem 5. Note also that the asymptotic efficieny of $S_{N,2}$ and $S_{N,Lischer}$ (and $S_{N,3}$ if $\sigma_1 = 0$ or $\sigma_0 = 0$) is the same as for the Q estimator of Rousseeuw and Croux (1992, 1993) for estimating the scale in a one-sample model. $S_{N,1}$ is also less efficient than $S_{N,2}$, but it reveals the advantages of less computational effort and a smaller gross error sensitivity (which has not been considered in this paper). Therefore the final choice between $S_{N,1}$ and $S_{N,2}$ should depend on the practical application.

Simulation results of $S_{N,3}$

	$\sigma_1 = 1$ $\sigma_0 = 0$		$\sigma_1 = 1$ $\sigma_0 = 1$		$\sigma_1 = 0$ $\sigma_0 = 1$	
	$N = 12$	$N = 30$	$N = 12$	$N = 30$	$N = 12$	$N = 30$
Std.Var.:	0.953	0.768	0.500	0.402	0.404	0.333
Mean:	1.060	1.020	1.444	1.429	1.018	1.011
Rel.Eff.:	57.3%	68.0%	68.5%	77.7%	64.8%	75.9%
Asymp.Rel.Eff.	82.2%		84.6%		82.2%	

The finite sample efficiency can be improved, in particular for $S_{N,3}$ with small N, if instead of the 0.25 quantile the quantile of the Q estimator of Rousseeuw and Croux (1992, 1993) is used. With this quantile we would get for example for $S_{N,3}$ at $\sigma_0 = 1 = \sigma_1$ and $N = 12$ an efficiency of 73.3% instead of 68.5%. The estimator of Lischer (1996) for σ_Y has in this case an efficiency of 64.6% although it uses the quantile of Rousseeuw and Croux. Moreover, for all σ_0 and σ_1 the asymptotic efficieny of the Lischer estimator for σ_Y is 82.26%.

The simulation study performed also included models with 3 and 4 replicates (not listed in the table). In all cases the efficiency of the robust estimators was even better as with $I = 2$ replicates. It can be concluded that $S_{N,2}$ and $S_{N,3}$ are promising estimators having high breakdown point and good finite sample efficiencies.

6. EXAMPLE

An interlaboratory comparison test was undertaken to assess the performance of $N = 23$ chemical laboratories with respect to the analysis of arsenic in soil. Each of the laboratories analyzed the arsenic content twice. The following table contains the ordered measurement results.

Lab	1	2	3	4	5	6	7	8	9	10	11	12
	0.19	1.60	1.90	2.00	2.00	2.00	2.00	2.00	2.02	2.10	2.10	2.44
	0.21	1.60	2.10	2.50	3.00	1.90	2.00	2.00	2.22	2.00	2.30	2.40

Lab	13	14	15	16	17	18	19	20	21	22	23
	2.50	2.70	2.74	3.80	4.00	4.80	6.20	7.00	8.00	8.98	27.00
	2.40	2.20	2.69	4.00	4.20	5.20	5.80	8.40	7.00	25.83	30.00

Approximately 60% of the laboratories are in a small range between 1.6 and 2.75, whereas some of the other laboratories provided results which are far from this interval. The reason for the considerable heterogenity of the data could be that the laboratories use different analytical methods or have different analytical equipment.

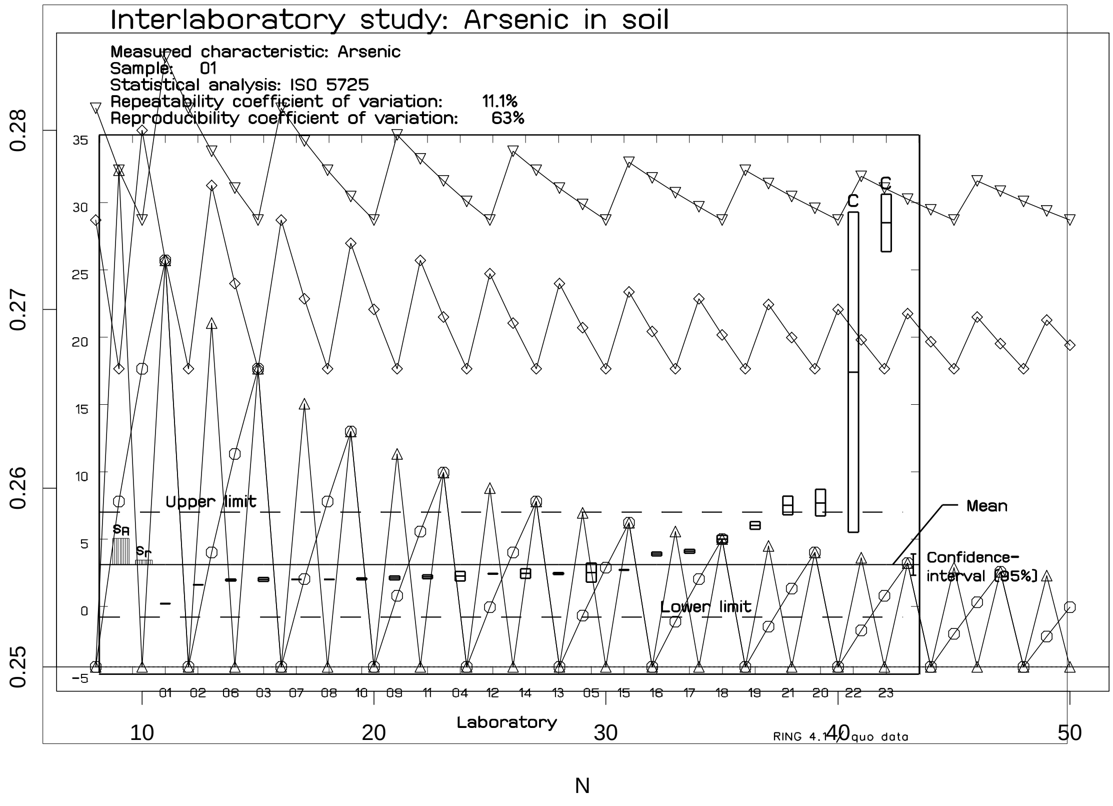

FIGURE 5

In a classical analysis according to ISO-5725-2 (1992) two outliers (lab 22 and 23) are detected. After elimination of these laboratories one obtains the estimates $\widehat{\sigma}_0 = 0.41$ and $\widehat{\sigma}_Y = 1.99$. Under the assumption of the normal distribution, the latter fits not very well to the fact, that the results of 60% of the laboratories are within an interval of length 1.15. See Figure 5.

A 'better' fit is provided by the new robust method: One obtains the estimate $S_{23,3} = 1.11$ for σ_Y. For σ_0 one obtains $S_{23,1} = 0.21$ and $S_{23,2} = 0.31$. The latter is higher because of the higher gross error sensitivity of $S_{N,2}$. But both estimates are smaller than the classical estimate according to ISO-5725-2.

Note that the results of ISO-5725-2 are highly depending on the results of the outlier tests. If in laboratory 20 the measurement value 8.40 would be replaced by 9.00, then the outlier identification procedure of ISO 5725-2 would provide five instead of two outlier laboratories. Then one obtains $\widehat{\sigma}_0 = 0.23$ and $\widehat{\sigma}_Y = 1.05$, which is very close to the results of the robust estimators.

7. APPENDIX

Proof of Theorem 1. To derive the block breakdown point assume that M blocks are corrupted. Then $H_{N,1}(0)$ is maximized if the corruptions in the M blocks provide equal observations in the M blocks. This implies that $M\,I\,(I-1)$ intra-block differences are equal 0 so that

$$H_{N,1}(0) = \frac{M\,I\,(I-1)}{N\,I\,(I-1)} < \frac{1}{2}$$

if and only if $M < \frac{N}{2}$. Hence we have no implosion if and only if $M \leq \left\lceil \frac{N-2}{2} \right\rceil$. The number of intra-block differences going to infinity is maximized if for example in the M corrupted blocks we have the following observations: $\overline{y}^v_{ni} = vi$, $n = 1, \ldots, M$, $i = 1, \ldots, I$ with $v \to \infty$. Then $M\,I\,(I-1)$ intra-block differences are going to infinity. Hence we have no explosion if and only if $\frac{M\,I\,(I-1)}{N\,I\,(I-1)} \leq \frac{1}{2}$, i.e. $M \leq \left\lfloor \frac{N}{2} \right\rfloor$.

To derive the measurement breakdown point we regard at first the implosion point. Let $M = L\,(I-1) + J$, $L = 0, \ldots, N$, $J = 0, \ldots, I-2$, be the number of corrupted observations. If in each block M_n observations, $M_n \in \{0, \ldots, I-1\}$, are corrupted so that $M_n + 1$ observations are equal then

$$(7.1) \quad \frac{1}{2}\sum_{n=1}^{N}(M_n+1)M_n = \frac{1}{2}\left(\sum_{n=1}^{N}(M_n)^2 + M\right)$$

intra-block differences are equal to zero. For fixed M the quantity (7.1) is maximized if $M_1 = \ldots = M_L = I-1$, $M_{L+1} = J$ and $M_{L+2} = \ldots = M_N = 0$. In particular for $J=0$ the quantity $H_{N,1}(0)$ is maximized if in L blocks we have $(I-1)$ corrupted observations so that all observations within each of the L blocks are equal. Now we determine the largest L so that $L\,(I-1)$ outlier cannot cause implosion. Then we have

$$H_{N,1}(0) = \frac{L\,I\,(I-1)}{N\,I\,(I-1)} < \frac{1}{2}$$

if and only if $L < \frac{N}{2}$. Hence, there is no implosion if and only if $L \leq \left\lceil \frac{N-2}{2} \right\rceil$. Because for $I \geq 2$

$$\lim_{N\to\infty} \frac{\left\lceil \frac{N-2}{2} \right\rceil \,(I-1)}{N\,I} = \frac{I-1}{2\,I} \geq \frac{1}{4}$$

the assertion holds for the implosion point.

Explosion: Let $M = N\,L + J$, $L = 0, \ldots, I$, $J = 0, \ldots, N-1$, be the number of outliers. If in each block M_n observations, $M_n \in \{0, \ldots, I\}$, are outlying then

$$(7.2) \quad \frac{1}{2}\sum_{n=1}^{N}(I-M_n)(I-M_n-1) = \frac{1}{2}\left(\sum_{n=1}^{N}(I-M_n)^2 - (N\,I - M)\right)$$

intra-block differences are not corrupted. The quantity (7.2) is minimized if $M_1 = \ldots = M_J = L+1$ and $M_{J+1} = \ldots = M_N = L$. At first we regard the case $J=0$ and determine the largest L so that $N\,L$ outliers cannot cause explosion. Then there is no explosion if and only if $\frac{N\,(I-L)\,(I-L-1)}{N\,I\,(I-1)} \geq \frac{1}{2}$, which is equivalent to $L \leq I - \frac{1}{2} - \frac{1}{2}\sqrt{2I^2 - 2I + 1}$. Hence the largest L without explosion is $L_0 = \left\lfloor I - \frac{1}{2} - \frac{1}{2}\sqrt{2I^2 - 2I + 1} \right\rfloor$. Now we determine the largest J so that $N\,L_0 + J$ outliers cannot cause explosion. In the presence of $N\,L_0 + J$ outliers at least

$$\frac{1}{2}\left(N(I-L_0)(I-L_0-1) - 2J(I-L_0-1)\right)$$

intra-class differences are not corrupted. There is no explosion if and only if

$$\frac{N(I-L_0)(I-L_0-1) - 2J(I-L_0-1)}{N\,I\,(I-1)} \geq \frac{1}{2},$$

which is equivalent to $J \leq \frac{N(I-L_0)}{2} - \frac{N\,I\,(I-1)}{4\,(I-L_0-1)}$. Therefore the largest J with no explosion is $J_0 = \left\lfloor \frac{N(I-L_0)}{2} - \frac{N\,I\,(I-1)}{4\,(I-L_0-1)} \right\rfloor$. Hence the explosion point is not smaller than

$$\begin{aligned} &\frac{N\,L_0 + J_0}{N\,I} \\ &\geq \frac{L_0}{I} + \frac{I-L_0}{2\,I} - \frac{I-1}{4\,(I-L_0-1)} - \frac{1}{N\,I} \\ (7.3) \quad &\xrightarrow{N\to\infty} \frac{I+L_0}{2\,I} - \frac{I-1}{4\,(I-L_0-1)} \geq \frac{1}{4}, \end{aligned}$$

where the inequality in (7.3) holds because it is equivalent to

(7.4) $\quad I\,L_0 \geq 2\,L_0^2 + 2\,L_0$.

For $I = 2, 3$, where $L_0 = 0$, the inequality (7.4) is trivially satified, and for $I \geq 4$ we always have $I \geq 2(I - \frac{1}{2} - \frac{1}{2}\sqrt{2I^2 - 2I + 1}) + 2$. □

Lemma 2. *Let be* $f(m_1, \dots, m_N) := \left(\sum_{n=1}^N m_n^2 - m\right)^2 - \sum_{n=1}^N m_n^2 (m_n - 1)^2$, *where* $m_n \in \{0, \dots, I\}$ *for* $n = 1, \dots, N$ *and* $m = \sum_{n=1}^N m_n$.

a) If $4\,I \leq m = L\,I + J$ *for* $L = 0, \dots, N$ *and* $J = 0, \dots, I - 1$, *then* $f(m_1, \dots, m_N)$ *is maximized if and and only if* $m_n = I$ *for* L *values,* $m_n = J$ *for one value and* $m_n = 0$ *for* $N - L - 1$ *values.*

b) If $N + I^2 + 1 \leq m = L\,N + J$ *for* $L = 0, \dots, I$ *and* $J = 0, \dots, N - 1$, *then* $f(m_1, \dots, m_N)$ *is minimized if and and only if* $m_n = L + 1$ *for* J *values and* $m_n = L$ *for* $N - J$ *values.*

Proof. Set $H := \sum_{n=3}^N m_n(m_n - 1)$. Then we have

$$f(m_1, \dots, m_N) = H^2 - \sum_{n=3}^{K} m_n^2 (m_n - 1)^2 + 2\,H\,(m_1(m_1 - 1) + m_2(m_2 - 1)) \\ + 2\,m_1(m_1 - 1) m_2 (m_2 - 1).$$

a) Assume without loss of generality $1 \leq m_1 \leq m_2 \leq \dots \leq m_K \leq I$, $m_{K+1} = \dots = m_N = 0$. If $m_2 < I$ then f is not maximized because we have $f(m_1, m_2, m_3, \dots, m_N) \leq f(m_1 - 1, m_2 + 1, m_3, \dots, m_N)$ if and only if

(7.5) $\quad H \geq m_2\,(m_1 - 1)$.

Because $m \geq 4\,I$ implies $K \geq 4$ and $H \geq 2 m_{K-1}(m_{K-1} - 1) \geq m_2(m_1 - 1)$ the inequality (7.5) holds and the inequality is strict if $m_{K-1} \geq 2$.

b) Assume without loss of generality $m_2 - m_1 \geq 2$ and $0 \leq m_1 \leq m_n$ for $n = 2, \dots, N$. Then we have $f(m_1, m_2, m_3, \dots, m_N) > f(m_1 + 1, m_2 - 1, m_3, \dots, m_N)$ if and only if

(7.6) $\quad H > m_1\,(m_2 - 1)$.

But (7.6) is satisfied because $N + I^2 + 1 \leq m$ implies $H \geq 2(I - 2)I + 2 > m_1(m_2 - 1)$. □

Proof of Theorem 2. To derive the block breakdown point assume that M blocks are corrupted. Then $H_{N,2}(0)$ is maximized if the corruptions in the M blocks provide equal observations in the M blocks. This implies that $\binom{M}{2}\,I^2\,(I - 1)^2$ intra-block differences are equal 0 so that

$$H_{N,2}(0) = \frac{\binom{M}{2}\,I^2\,(I - 1)^2}{\binom{N}{2}\,I^2\,(I - 1)^2} < \frac{1}{4}$$

if and only if $M < \frac{1}{2} + \frac{1}{2}\sqrt{N^2 - N + 1}$. Hence we have no implosion if and only if $M \leq \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{N^2 - N + 1} \right\rceil$. The number of second order differences going to infinity is maximized if for example in the M corrupted blocks we have the following observations: $\overline{y}^{v}_{ni} = v\,n\,i\,, n = 1, \ldots, M, i = 1, \ldots, I$ with $v \to \infty$. Then $\binom{N-M}{2} I^2\,(I-1)^2$ second order differences are not corrupted. Hence, we have no explosion if and only if $\frac{\binom{N-M}{2} I^2\,(I-1)^2}{\binom{N}{2} I^2\,(I-1)^2} \geq \frac{1}{4}$, i.e. $M \leq \left\lfloor N - \frac{1}{2} - \frac{1}{2}\sqrt{N^2 - N + 1} \right\rfloor$.

To derive the measurement breakdown point we regard at first the implosion point. If $M = L\,(I-1) + J$, $J = 0, \ldots, I-2$, observations are corrupted so that in each block $m_n := M_n + 1$ observations are equal ($M = \sum_{n=1}^{N} M_n$) then

$$(7.7) \quad \frac{1}{4}\left(\sum_{n=1}^{N} m_n^2 - M - N\right)^2 - \frac{1}{4}\sum_{n=1}^{N} m_n^2 (m_n - 1)^2$$

second order differences are equal to zero. According to Lemma 2, for fixed $m = M + N$ the quantity (7.7) is maximized if $M_1 = \ldots = M_L = I - 1$, $M_{L+1} = J$ and $M_{L+2} = \ldots = M_N = 0$. In particular for $J = 0$ at most $\frac{1}{2}\binom{L}{2}\, I\,(I-1)\,I\,(I-1)$ second order differences are equal to 0 so that

$$H_{N,2}(0) = \frac{\binom{L}{2}\, I\,(I-1)\,I\,(I-1)}{\binom{N}{2}\, I\,(I-1)\,I\,(I-1)} < \frac{1}{4}$$

if and only if $L < \frac{1}{2} + \frac{1}{2}\sqrt{N^2 - N + 1}$. Hence there is no implosion if and only if $L \leq \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{N^2 - N + 1} \right\rceil$. Because for $I \geq 2$

$$\lim_{N \to \infty} \frac{(I-1)\,\left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{N^2 - N + 1} \right\rceil}{N\,I} = \frac{I-1}{2\,I} \geq \frac{1}{4}$$

the assertion holds for the implosion.

Explosion: If $M = N\,L + J$, $J = 0, \ldots, N-1$, observations are going to infinity, then in each block $m_n = I - M_n$ observations are not corrupted, where $M = \sum_{n=1}^{N} M_n$. This implies that

$$(7.8) \quad \frac{1}{4}\left(\sum_{n=1}^{N} m_n^2 - (N\,I - M)\right)^2 - \frac{1}{4}\sum_{n=1}^{N} m_n^2 (m_n - 1)^2$$

second order differences are not corrupted. According to Lemma 2, for fixed $m = NI - M$ the quantity (7.8) is minimized if $M_1 = \ldots = M_J = L + 1$ and $M_{J+1} = \ldots = M_N = L$. At first we regard the case $J = 0$ and determine the largest L so that $N\,L$ outliers cannot cause explosion. Then at least $\frac{1}{2}\binom{N}{2}\,(I-L)\,(I-L-1)\,(I-L)\,(I-L-1)$ second order differences are not corrupted. There is no explosion if and only if

$$\frac{\binom{N}{2}\,(I-L)\,(I-L-1)\,(I-L)\,(I-L-1)}{\binom{N}{2}\, I\,(I-1)\,I\,(I-1)} \geq \frac{1}{4},$$

which is equivalent with $L \leq I - \frac{1}{2} - \frac{1}{2}\sqrt{2I^2 - 2I + 1}$. Therefore the largest L without explosion is $L_0 = \left\lfloor I - \frac{1}{2} - \frac{1}{2}\sqrt{2I^2 - 2I + 1}\right\rfloor$. Now we determine the largest J so that $N\,L_0 + J$ outliers cannot cause explosion. Set $L_1 = I - L_0$. In the presence of $N\,L_0 + J$ outliers at least

$$\begin{aligned}
&\frac{1}{2}\binom{J}{2}(L_1-1)^2(L_1-2)^2 + \frac{1}{2}\binom{N-J}{2}L_1^2(L_1-1)^2 \\
&\quad + \frac{1}{2}J(N-J)(L_1-1)(L_1-2)\,L_1(L_1-1) \\
&= \frac{1}{4}(L_1-1)^2\left(J^2 4 + J\,(4L_1 - 4 - 4N\,L_1) + (N^2-N)\,L_1^2\right)
\end{aligned}$$

second order differences are not corrupted. There is no explosion if and only if

$$\frac{\frac{1}{4}(L_1-1)^2\,(J^2 4 + J\,(4L_1 - 4 - 4N\,L_1) + (N^2-N)\,L_1^2)}{\frac{1}{4}N\,(N-1)\,I^2\,(I-1)^2} \geq \frac{1}{4},$$

which is equivalent to

$$\begin{aligned}
J \leq J_1 := {} & \frac{N\,L_1 + 1 - L_1}{2} \\
& - \sqrt{\frac{(N\,L_1 + 1 - L_1)^2}{4} - \frac{(N^2-N)L_1^2}{4} + \frac{(N^2-N)I^2(I-1)^2}{16\,(L_1-1)^2}}.
\end{aligned}$$

Therefore the largest J with no explosion is $J_0 = \lfloor J_1 \rfloor$. Hence the explosion point is not smaller than

$$\begin{aligned}
& \frac{N\,L_0 + J_0}{N\,I} \\
& \xrightarrow{N\to\infty} \frac{L_0}{I} + \frac{L_1}{2\,I} - \frac{1}{I}\sqrt{\frac{L_1^2}{4} - \frac{L_1^2}{4} + \frac{I^2(I-1)^2}{16\,(L_1-1)^2}} \\
& = \frac{L_0}{I} + \frac{I - L_0}{2\,I} - \frac{I-1}{4\,(I - L_0 - 1)} \geq \frac{1}{4}, \qquad (7.9)
\end{aligned}$$

where the validity of the inequality in (7.9) was already shown in the proof of Theorem 1. □

Proof of Theorem 3. To derive the block breakdown point assume that M blocks are corrupted. Then $H_{N,3}(0)$ is maximized if the corruptions in the M blocks provide equal observations in the M blocks. This implies that $\binom{M}{2}\,I^2$ inter-block differences are equal 0 so that

$$H_{N,3}(0) = \frac{\binom{M}{2}\,I^2}{\binom{N}{2}\,I^2} < \frac{1}{4}$$

if and only if $M < \frac{1}{2} + \frac{1}{2}\sqrt{N^2 - N + 1}$. Hence we have no implosion if and only if $M \leq \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{N^2 - N + 1}\right\rceil$. The number of second order differences going to infinity is maximized if for example in the M corrupted blocks we have the following observations:

$\overline{y}^v_{ni} = v\,n\,i$, $n = 1, \dots, M$, $i = 1, \dots, I$ with $v \to \infty$. Then $\binom{N-M}{2} I^2$ inter-block differences are not corrupted. Hence we have no explosion if and only if $\frac{\binom{N-M}{2} I^2}{\binom{N}{2} I^2} \ge \frac{1}{4}$, i.e. $M \le \left\lfloor N - \frac{1}{2} - \frac{1}{2}\sqrt{N^2 - N + 1} \right\rfloor$.

To derive the measurement breakdown point we regard at first the implosion point. Let $M = N\,L - 1 + J$, $L = 0, \dots, I$, $J = 0, \dots, N-1$, be the number of observations which are corrupted so that $M+1$ observations have equal values. If in each block M_n observations, $M_n \in \{0, \dots, I\}$, are equal to a fixed value, then

$$\frac{1}{2}\sum_{n=1}^{N} M_n(M + 1 - M_n) = \frac{(M+1)^2}{2} - \frac{1}{2}\sum_{n=1}^{N} M_n^2 \tag{7.10}$$

inter-block differences are equal to zero. For fixed M the quantity (7.10) is maximized if $M_1 = \ldots = M_J = L + 1$ and $M_{J+1} = \ldots = M_N = L$. In particular for $J = 0$ we obtain that at most $\binom{N}{2} L^2$ inter-block differences are equal to zero. Hence, for $J = 0$, we have no implosion if and only if

$$H_{N,3}(0) = \frac{\binom{N}{2} L^2}{\binom{N}{2} I^2} < \frac{1}{4},$$

which is equivalent to $L < \frac{I}{2}$. I.e. the largest L with no implosion is $L_0 = \left\lceil \frac{I}{2} \right\rceil - 1$. Now we determine the largest J so that $N\,L_0 - 1 + J$ corrupted observations cannot cause implosion. In the presence of $N\,L_0 - 1 + J$ corrupted observations at most

$$\frac{1}{2}\left(J^2 + J(2NL_0 - 2L_0 - 1) + (N^2 - N)L_0^2\right)$$

inter-block differences are equal to zero. There is no implosion if and only if

$$\frac{\frac{1}{2}\left(J^2 + J(2NL_0 - 2L_0 - 1) + (N^2 - N)L_0^2\right)}{\frac{1}{2}\,N\,(N-1)\,I^2} < \frac{1}{4},$$

which is equivalent to

$$J < J_1 := \frac{-2N\,L_0 + 2L_0 + 1}{2} + \sqrt{\frac{(2N\,L_0 - 2L_0 - 1)^2}{4} + \frac{(N^2 - N)}{4}\,(I^2 - 4L_0^2)}.$$

Therefore, the largest J with no implosion is $J_0 := \lceil J_1 \rceil - 1$. Hence, the implosion point is not smaller than

$$\frac{N\,L_0 + J_0}{N\,I} \xrightarrow{N \to \infty} \frac{L_0}{I} - \frac{L_0}{I} + \frac{1}{I}\sqrt{\frac{4L_0^2}{4} + \frac{I^2 - 4L_0^2}{4}} = \frac{1}{2}.$$

Explosion: Let $M = I\,L + J$, $L = 0, \dots, N$, $J = 0, \dots, I-1$, be the number of outliers. If in each block M_n observations, $M_n \in \{0, \dots, I\}$, are outlying then

$$\frac{1}{2}\sum_{n=1}^{N}(I - M_n)(N\,I - M + M_n - I) = \frac{(N\,I - M)^2}{2} - \frac{1}{2}\sum_{n=1}^{N}(I - M_n)^2 \tag{7.11}$$

inter-block differences are not corrupted. For fixed M the quantity (7.11) is minimized if $M_1 = \ldots = M_L = I$, $M_{L+1} = J$ and $M_{L+1} = \ldots = M_N = 0$. In particular for $J = 0$ we have no explosion if and only if

$$\frac{\binom{N-L}{2} I^2}{\binom{N}{2} I^2} \geq \frac{1}{4}$$

which is equivalent to $L \leq N - \frac{1}{2} - \frac{1}{2}\sqrt{N^2 - N + 1}$, i.e. $L \leq \left\lfloor N - \frac{1}{2} - \frac{1}{2}\sqrt{N^2 - N + 1} \right\rfloor$.
$\square$

To derive the asymptotic distribution of the estimators we firstly show a general result which extends results of Ghosh (1971) and Serfling (1984) concerning generalized L-statistics. For that assume that we have independent and identically distributed random vectors $Y_1, Y_2, Y_3, \ldots$ with distribution function F and dimension I and functions $h_l : I\!R^{l \times s} \rightarrow I\!R$ for $l = 1, \ldots, L$ which satisfy

(7.12) $H_F(y) := \mathrm{P}(h_1(Y_1, \ldots, Y_s) \leq y) = \mathrm{P}(h_l(Y_{n_1}, \ldots, Y_{n_s}) \leq y)$

for all $l = 1, \ldots, L, y \in I\!R, n_1 > \ldots > n_s$. Define

$$U(Y_{n_1}, \ldots, Y_{n_s}, y) := \frac{1}{L} \sum_{l=1}^{L} 1\{h_l(Y_{n_1}, \ldots, Y_{n_s}) \leq y\},$$

and

$$H_N(y) := \frac{1}{\begin{pmatrix} N \\ s \end{pmatrix}} \sum_{1 \leq n_1 < n_2 < \ldots < n_s \leq N} U(Y_{n_1}, \ldots, Y_{n_s}, y),$$

and assume that

(7.13) $U(y_{\pi_1}, \ldots, y_{\pi_s}, y) = U(y_1, \ldots, y_s, y)$

for all permutations $\pi : \{1, \ldots, s\} \rightarrow \{1, \ldots, s\}$ and all $y \in I\!R$, $y_1, \ldots, y_s \in I\!R^I$.

Note that the estimators $S_{N,1}$, $S_{N,2}$ and $S_{N,3}$ are of the form $c \cdot H_N^{-1}(p)$, where c is a constant and the conditions (7.12) and (7.13) are satisfied.

To derive the asymptotic normality of estimators of the form $H_N^{-1}(p)$ we use the fact that $H_N(y)$ is a U-statistic with kernel $\psi(y_{n_1}, \ldots, y_{n_s}) = U(y_{n_1}, \ldots, y_{n_s}, y)$. Because of (7.13) the kernel is symmetric. Then with the theorem of Hoeffding (see e.g. Serfling 1980, p. 192) we have

(7.14) $\mathcal{L}(\sqrt{N}(H_N(y) - \gamma)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, s^2 \sigma_1^2)$,

where

$$\gamma := \mathrm{E}(U(Y_{n_1}, \ldots, Y_{n_s}, y)) = H_F(y)$$

and $\sigma_1^2 := \mathrm{Var}(\psi_y^1(Y_1)$ with

$$\psi_y^1(y_1) := \mathrm{E}(U(Y_1, ..., Y_s, y)|Y_1 = y_1)$$
$$= \frac{1}{L}\sum_{l=1}^{L} \mathrm{P}(h_l(Y_1, ..., Y_s) \le y|Y_1 = y_1).$$

Using (7.14) for $y = H_F^{-1}(p)$ the following lemma can be proved.

Lemma 3. *If $H_F'(H_F^{-1}(p)) > 0$ exists then*

$$\mathcal{L}(\sqrt{N}(H_N^{-1}(p) - H_F^{-1}(p))) \overset{N\to\infty}{\longrightarrow} \mathcal{N}\left(0, \frac{s^2\sigma_1^2}{(H_F'(H_F^{-1}(p)))^2}\right),$$

where $\sigma_1^2 := Var(\psi^1(Y_1)$ with

$$\psi^1(y_1) := \frac{1}{L}\sum_{l=1}^{L} P(h_l(Y_1, ..., Y_s) \le H_F^{-1}(p)|Y_1 = y_1)$$

Proof. Because of (7.14) as in Serfling (1984) we have only to show that

$$\sqrt{N}\, R_N \to 0 \quad \text{in probability},$$

where

$$R_N := H_N^{-1}(p) - H_F^{-1}(p) - \frac{p - H_N(H_F^{-1}(p))}{H_F'(H_F^{-1}(p))}.$$

This can be shown along the lines of the proof of Theorem 1 in Ghosh (1971) as Serfling (1984), p. 84, found out for the case $L = 1$. □

Proof of Theorem 4. Note that

$$S_{N,1}(y_1, \ldots, y_N) = \frac{1}{\sqrt{2}\Phi^{-1}((p+1)/2)}\, H_{N,1}^{-1}(p)$$

with $p = \frac{1}{2}$. To show that the proof also holds for general $p \in (0, 1)$ we will work with p instead of $\frac{1}{2}$. Set

$$H_1(y) := \mathrm{P}(|Y_{11} - Y_{12}| \le y) = \mathrm{P}(|Z_{11} - Z_{12}| \le y) = 2\Phi\left(\frac{y}{\sqrt{2}\sigma_0}\right) - 1$$

and

$$h_{ij}(y_1) := |y_{1i} - y_{1j}| \ \text{ for } \ i \ne j.$$

Then we have $H_1^{-1}(p) = \sqrt{2}\sigma_0\Phi^{-1}((p+1)/2)$ and $H_1'(y) = \frac{\sqrt{2}}{\sigma_0}\phi\left(\frac{y}{\sqrt{2}\sigma_0}\right)$. Hence, Lemma 3 with

$$\psi^1(y_1) = \frac{1}{I(I-1)}\sum_{i\ne j} 1\{|y_{1i} - y_{1j}| \le H_1^{-1}(p)\}$$

provides that

$$\sqrt{N}(S_{N,1} - \sigma_0) = \sqrt{N}\ \left(\frac{1}{\sqrt{2}\Phi^{-1}((p+1)/2)}H_{N,1}^{-1}(p) - \frac{1}{\sqrt{2}\Phi^{-1}((p+1)/2)}H_1^{-1}(p)\right)$$

is asymptotically normal with mean 0 and variance

$$\left(\frac{1}{\sqrt{2}\Phi^{-1}((p+1)/2)}\right)^2\left(\frac{1}{\frac{\sqrt{2}}{\sigma_0}\phi(\Phi^{-1}((p+1)/2)}\right)^2\mathrm{Var}(\psi^1(Y_1)).$$

Because

$$\begin{aligned}&\mathrm{P}(|Y_{11}-Y_{12}|\le y,|Y_{12}-Y_{13}|\le y|Z_{12}=z)\\&=\left(\Phi\left(\frac{z+y}{\sigma_0}\right)-\Phi\left(\frac{z-y}{\sigma_0}\right)\right)^2\end{aligned}$$

we have for $y=H_1^{-1}(p)=\sqrt{2}\sigma_0\Phi^{-1}((p+1)/2)$

$$\begin{aligned}&v_1(p):=\mathrm{P}(|Y_{11}-Y_{12}|\le y,|Y_{12}-Y_{13}|\le y)=\\&=\int\left[\Phi\left(z+\sqrt{2}\Phi^{-1}((p+1)/2)\right)-\Phi\left(z-\sqrt{2}\Phi^{-1}((p+1)/2)\right)\right]^2\phi(z)dz.\end{aligned}$$

Moreover, for $y=H_1^{-1}(p)$ we have

$$\mathrm{P}(|Y_{11}-Y_{12}|\le y)=2\Phi\left(\Phi^{-1}((p+1)/2)\right)-1=p.$$

This implies

$$\begin{aligned}&\mathrm{Cov}\left(1\{|Y_{11}-Y_{12}|\le H_1^{-1}(p)\},1\{|Y_{12}-Y_{13}|\le H_1^{-1}(p)\}\right)\\&=v_1(p)-p^2\end{aligned}$$

so that

$$\begin{aligned}&\mathrm{Var}(\psi^1(Y_1))\\&=\frac{2}{I(I-1)}\mathrm{Var}\left(1\{|Y_{11}-Y_{12}|\le H_1^{-1}(p)\}\right)\\&\quad+\frac{4(I-2)}{I(I-1)}\mathrm{Cov}\left(1\{|Y_{11}-Y_{12}|\le H_1^{-1}(p)\},1\{|Y_{12}-Y_{13}|\le H_1^{-1}(p)\}\right)\\&=\frac{2}{I(I-1)}p(1-p)+\frac{4(I-2)}{I(I-1)}\left(v_1(p)-p^2\right)\end{aligned}$$

provides the assertion with $v_1(p)=v_1$ if $p=\frac{1}{2}$. □

Proof of Theorem 5. Note that

$$S_{N,2}(y_1,\dots,y_N)=\frac{1}{2\Phi^{-1}((p+1)/2)}H_{N,2}^{-1}(p)$$

with $p=\frac{1}{4}$. Again we will work with p instead of $\frac{1}{4}$. Set

$$H_2(y):=\mathrm{P}\left(|Y_{11}-Y_{12}-Y_{21}+Y_{22}|\le y\right)=2\Phi\left(\frac{y}{2\sigma_0}\right)-1$$

and

$$h_{ijkl}(y_1,y_2):=|y_{1i}-y_{1j}-y_{2k}+y_{2l}|\ \text{ for } i\ne j,\ k\ne l.$$

Then we have $H_2^{-1}(p) = 2\sigma_0 \Phi^{-1}((p+1)/2)$, $H_2'(y) = \frac{1}{\sigma_0}\phi\left(\frac{y}{2\sigma_0}\right)$, and

$$\begin{aligned}\psi^1(y_1) &:= \frac{1}{I^2(I-1)^2}\sum_{i\neq j}\sum_{k\neq l}\mathrm{P}(h_{ijkl}(Y_1,Y_2) \leq H_2^{-1}(p)|Y_1 = y_1)\\ &= \frac{1}{I(I-1)}\sum_{i\neq j}\left[\Phi\left(\frac{-y_{1i}+y_{1j}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right) - \Phi\left(\frac{-y_{1i}+y_{1j}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right].\end{aligned}$$

Hence, Lemma 3 provides that

$$\sqrt{N}(S_{N,2}-\sigma_0) = \sqrt{N}\,\left(\frac{1}{2\Phi^{-1}((p+1)/2)}H_{N,2}^{-1}(p) - \frac{1}{2\Phi^{-1}((p+1)/2)}H_2^{-1}(p)\right)$$

is asymptotically normal with mean 0 and variance

$$4\,\left(\frac{1}{2\Phi^{-1}((p+1)/2)}\right)^2\left(\frac{1}{\frac{1}{\sigma_0}\phi(\Phi^{-1}((p+1)/2)}\right)^2\mathrm{Var}(\psi^1(Y_1)).$$

To calculate $\mathrm{Var}(\psi^1(Y_1))$ at first note that for $i \neq j$

$$\begin{aligned}&v_1(p) := \mathrm{E}\left(\Phi\left(\frac{-Y_{1i}+Y_{1j}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right) - \Phi\left(\frac{-Y_{1i}+Y_{1j}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right)\\ &= \int\left[\Phi\left(z+\sqrt{2}\Phi^{-1}((p+1)/2)\right) - \Phi\left(z-\sqrt{2}\Phi^{-1}((p+1)/2)\right)\right]\phi(z)\\ &= p\end{aligned}$$

and

$$\begin{aligned}&v_2(p) := \mathrm{E}\left(\left[\Phi\left(\frac{-Y_{1i}+Y_{1j}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right) - \Phi\left(\frac{-Y_{1i}+Y_{1j}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right]^2\right)\\ &= \int\left[\Phi\left(z+\sqrt{2}\Phi^{-1}((p+1)/2)\right) - \Phi\left(z-\sqrt{2}\Phi^{-1}((p+1)/2)\right)\right]^2\phi(z)dz.\end{aligned}$$

Because

$$\begin{aligned}&\mathrm{E}\left(\left[\Phi\left(\frac{-Y_{11}+Y_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right) - \Phi\left(\frac{-Y_{11}+Y_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right]\right.\\ &\quad\left.\cdot\left[\Phi\left(\frac{-Y_{13}+Y_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right) - \Phi\left(\frac{-Y_{13}+Y_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right]\middle| Z_{12}=z_{12}\right)\\ &= \mathrm{E}\left(\Phi\left(\frac{-Z_{11}+z_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right) - \Phi\left(\frac{-Z_{11}+z_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right)\\ &\quad\cdot\mathrm{E}\left(\Phi\left(\frac{-Z_{13}+z_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right) - \Phi\left(\frac{-Z_{13}+z_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right)\end{aligned}$$

we have

$$\begin{aligned} v_3(p) := \mathrm{E}\Bigg(&\left[\Phi\left(\frac{-Y_{11}+Y_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)-\Phi\left(\frac{-Y_{11}+Y_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right] \\ &\cdot\left[\Phi\left(\frac{-Y_{13}+Y_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)-\Phi\left(\frac{-Y_{13}+Y_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right]\Bigg) \\ = \int\Bigg[\int&\Big[\Phi\left(z_2/\sqrt{2}-z_1/\sqrt{2}+\sqrt{2}\Phi^{-1}((p+1)/2)\right) \\ &-\Phi\left(z_2/\sqrt{2}-z_1/\sqrt{2}-\sqrt{2}\Phi^{-1}((p+1)/2)\right)\Big]\phi(z_1)dz_1\Bigg]^2\phi(z_2)dz_2. \end{aligned}$$

This and the fact that $\Phi(x+y)-\Phi(x-y)=\Phi(-x+y)-\Phi(-x-y)$, for all $x,y\in I\!R$, implies

$$\begin{aligned} &\mathrm{Var}(\psi^1(Y_1)) \\ &= \frac{2}{I(I-1)}\mathrm{Var}\left(\Phi\left(\frac{-Y_{11}+Y_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)-\Phi\left(\frac{-Y_{11}+Y_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right) \\ &\quad+\frac{4(I-2)}{I(I-1)}\mathrm{Cov}\left(\Phi\left(\frac{-Y_{11}+Y_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)-\Phi\left(\frac{-Y_{11}+Y_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right),\right. \\ &\qquad\qquad\left.\Phi\left(\frac{-Y_{13}+Y_{12}+H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)-\Phi\left(\frac{-Y_{13}+Y_{12}-H_2^{-1}(p)}{\sqrt{2}\sigma_0}\right)\right) \\ &= \frac{2}{I(I-1)}\left(v_2(p)-p^2\right)+\frac{4(I-2)}{I(I-1)}\left(v_3(p)-p^2\right) \end{aligned}$$

so that the assertion follows with $v_2=v_2(p)$ and $v_3=v_3(p)$. □

Proof of Theorem 6. Note that

$$S_{N,3}(y_1,\ldots,y_N)=\frac{1}{2\Phi^{-1}((p+1)/2)}H_{N,3}^{-1}(p)$$

with $p=\frac{1}{4}$. Again we will work with p instead of $\frac{1}{4}$. Set

$$\begin{aligned} H_3(y) &:= \mathrm{P}(|Y_{11}-Y_{21}|\le y) \\ &= \mathrm{P}(|X_1+Z_{11}-X_2-Z_{21}|\le y)=2\Phi\left(\frac{y}{\sqrt{2\sigma_1^2+2\sigma_0^2}}\right)-1 \end{aligned}$$

and

$$h_{ij}(y_1,y_2):=|y_{1i}-y_{2j}| \quad\text{for}\quad i,j=1,\ldots,I.$$

Then we have $H_3^{-1}(p) = \sqrt{2\sigma_1^2 + 2\sigma_0^2}\Phi^{-1}((p+1)/2)$, $H_3'(y) = \frac{\sqrt{2}}{\sqrt{\sigma_1^2+\sigma_0^2}}\phi\left(\frac{y}{\sqrt{2\sigma_1^2+2\sigma_0^2}}\right)$, and

$$\begin{aligned}\psi^1(y_1) &:= \frac{1}{I^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\mathrm{P}\left(h_{ij}(Y_1,Y_2) \le H_3^{-1}(p)|Y^1 = y_1\right)\\ &= \frac{1}{I}\sum_{i=1}^{I}\left[\Phi\left(\frac{y_{1i}}{\sqrt{\sigma_1^2+\sigma_0^2}} + \sqrt{2}\Phi^{-1}((p+1)/2)\right)\right.\\ &\qquad\qquad \left. -\ \Phi\left(\frac{y_{1i}}{\sqrt{\sigma_1^2+\sigma_0^2}} - \sqrt{2}\Phi^{-1}((p+1)/2)\right)\right].\end{aligned}$$

Hence, Lemma 3 provides that

$$\begin{aligned}&\sqrt{N}\left(S_{N,3} - \sqrt{\sigma_1^2+\sigma_0^2}\right)\\ &= \sqrt{N}\left(\frac{1}{\sqrt{2}\Phi^{-1}((p+1)/2)}H_{N,3}^{-1}(p) - \frac{1}{\sqrt{2}\Phi^{-1}((p+1)/2)}H_3^{-1}(p)\right)\end{aligned}$$

is asymptotically normal with mean 0 and variance

$$4\left(\frac{1}{\sqrt{2}\Phi^{-1}((p+1)/2)}\right)^2\left(\frac{1}{\frac{\sqrt{2}}{\sqrt{\sigma_1^2+\sigma_0^2}}\phi(\Phi^{-1}((p+1)/2)}\right)^2\mathrm{Var}(\psi^1(Y_1)).\square$$

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