On the rate of convergence to normality of estimates of regression coefficient for fields of associated random variables

T.V. Koval[1] and V. Kurotschka[2] [1]Diese Arbeit entstand während eines durch den Deutschen Akademischen Austauschdienstes geförderten Aufenthaltes an der Freien Universität Berlin.

Abstract. The rate of convergence in the central limit theorem is studied for the least square estimates of regression coefficients of associated random fields.

1 Introduction

The rate of convergence in the central limit theorem for fields of weakly dependent random variables has been studied in many papers (see, for instance, Tikhomirov (1980) , Synklodas (1986) Koval (1992)). Literature on this subject was reviewed by Eberling (1986). Conditions of m-dependence and strong mixing are practically difficult to verify, i.e., to justify in practical situations.

One of the more practically applicable approach to modelling weakly dependent observations has been introduced by Esyry, Proschan and Walkup (1967) described by the concept of association or positive dependence. Recall that a finite collection $\xi = (\xi_i, \dots \xi_m)$ of real-valued random variables is called associated if, for any coordinatewise nondecreasing functions $f, g : R^m \to R$ such that $Ef(\xi)g(\xi), Ef(\xi)$ and $Eg(\xi)$ are finite, one has

$$\text{cov}\ (f(\xi), g(\xi)) \geq 0.$$

For an infinite number of real-valued random variables to be associated means that all finite subfamilies have this property. A family of independent random variables is associated (see Esary, Proschan and Walkup (1967)). Moreover, if $\eta = (\eta_1, \dots, \eta_m)$ is a collection (in particular a singleton) of associated variables and $h_K : R^m \to R$, $k = 1, \dots, n$ are coordinate-wise nondecreasing functions, then $h_1(\eta), \dots, h_n(\eta)$ are associated (see the same).

The central limit theorem for strictly stationary fields of associated random variables on Z^d has been proved by Newman (1980). Rates of convergence in the central limit theorem for associated random variables have been obtained by Birkel (1988) and Wood (1983), and for fields by Bulinskii (1995).

Consider a random field

$$\xi_j = \theta\varphi_j + \epsilon_j$$

where $\varphi_j : Z^d \to R^1$ is a known function, $\epsilon_j : Z^d \to R^1$ is a random field with mean 0. Let a system of bounded increasing sets A_n be singled out in Z^d and let θ be an unknown parameter to be estimated from observations of the random field ϵ_j on the sets A_n.

$$\text{Let } \widehat{\theta}_n = \sum\nolimits_{j\in A_n} \xi_j\varphi_j/d_n^2$$

be the least square estimates for the parameter θ where $d_n^2 = \sum_{j\in A_n}\varphi_j^2$. Obviously, $\widehat{\theta}_n$ is an unbiased estimate for θ. The theorem that establishes the asymptotic normality of least square estimate for the regression coefficient of associated random fields was proved by Koval (1996). In this paper, the rate covergence in the central limit theorem is studied for such estimates.

2 The results

Theorem. Let a field $\{\epsilon_j : j \in Z^d\}$ of associated random variables with zero mean satisfiy the following assumptions:

1) $\sup_j E|\epsilon_j|^S < \infty$ for some $s \in (2,3]$

2) $u(n) = 0(e^{-\lambda n})$ as $n \to \infty$ for some $\lambda > 0$

3) $\varphi_j \geq 0$, $\sup_j \frac{\varphi_j}{d_n} \leq \frac{L}{|A_n|^{\frac{1}{2}}}$

4) $\lim_{n\to\infty} \frac{\sigma_n^2}{d_n^2} = \sigma_0^2$

Then

$$\Delta_n = \sup_x |P(dn(\widehat{\theta}_n - \theta) < x) - \varphi(x)| \leq B((ln|A_n|)^{d(s-1)}|A_n|^{-\frac{s-2}{2}} + |1 - \frac{\sigma_n^2}{d_n^2\sigma_0^2}|) \quad (2.1)$$

where B being independent of $|An|$, positive constant,

$$u(n) = \sup_{j\in Z^d} \sum\nolimits_{q\in Z^d:|q-j|\geq n} cov(\epsilon_j, \epsilon_q),$$

$|y| = \max_{|\leq K\leq d} |y_K|$ for $y \in R^d$, $|A_n|$ is the number of points in A_n, $\sigma_n^2 = E(\sum_{j\in A_n}\epsilon_j\varphi_j)^2$, $\varphi(x)$-denotes distribution function of normal law with $N(0,\sigma_0^2)$

The theorem is proved by using the method of charactristic functions and ideas Birkel (1988).

3 Proof.

Let us introduce the notation $A^{(0)}_{j,n} = A_n, A^{(\ell)}_{j,n} = \{q \in A_n : |q-j| > \ell m\}$, $V^{(\ell)}_{j,n} = A^{(\ell-1)}_{j,n}$, $x_j = \epsilon\varphi_j/(d_n\varphi_0)$

$$S^{(0)}_{j,n} = S_n = \sum\nolimits_{j\in A_n} x_j, S^{(e)}_{j,n} = \sum\nolimits_{k\in A^{(e)}_{j,n}} x_k.$$

$$U^{(e)}_{j,n} = \sum\nolimits_{k\in V^{(e)}_{j,n}} x_k, \xi^{(e)}_j = \exp\{\text{it } U^{(e)}_{j,n}\} - 1$$

$$\xi^{(r)}_j = x_j \Pi^r_{e=1}\xi^{(e)}_j, \quad f_n(t) = E\exp\{\text{it } S_n\}$$

Then $f_n'(t) = i\sum\limits_{j\in A_n} Ex_j \exp\{\text{it } S^{(0)}_{j,n}\}$

Using the results from Tikhomirov (1980) we arrive at the representation

$$f_n'(t) = i\sum\nolimits_{j\in A_n} Ex_j \exp\{\text{it } S^{(r)}_{j,n}\}+$$

$$+i\sum\nolimits_{r=1}^k \sum\nolimits_{j\in A_n} E\zeta^{(r)}_j(\exp\{\text{it } S^{(r+1)}_{j,n}\} - f_n(t)) + if_n(t)\sum\nolimits_{r=1}^k \sum\nolimits_{j\in A_n} E\zeta^{(n)}_j +$$

$$+ i\sum\nolimits_{j\in A_n} E\zeta^{(k+1)}_j \exp\{\text{it } S^{(k+1)}_{j,n}\} \tag{3.2}$$

Since in the analysis of (2) an important role is played by Lemma 3.1 (see Birkel (1988)), we include it here.

Lemma (Birkel (1988)) let A and B finite sets and let $x_j, j \in A\cup B$, be associated random variables. Let functions $f : R^{|A|} \to R, g : R^{|B|} \to R$ are partially differentiable with bounded partial derivatives, then

$$|cov(f((x_i)_{i\in A}), g((x_j)_{j\in B}))| \le \sum\nolimits_{i\in A}\sum\nolimits_{j\in B} ||\frac{\partial f}{\partial t_i}||_\infty ||\frac{\partial g}{\partial t_j}||_\infty cov(x_i, x_j) \tag{3.3}$$

Denoting $c_s = c_s(A_n) = \max_{j\in A_n}(E|\epsilon_j|^s)^{\frac{1}{s}}$

$$b_s(m,r) = b_s(A_n,m,r) = \max\nolimits_{j\in A_n}\max\nolimits_{1\le\ell\le r}(E|U^{(e)}_{j,n}|^s)^{\frac{1}{s}} \tag{3.4}$$

Denotes $\theta_i(t)$ a complex-valued function with the modulus bounded by 1 one obtains.

Lemma 1 For all $t \in R$

$$i\sum_{j\in A_n} Ex_j \exp\{\text{it } S^{(1)}_{j,n}\} = 2A\theta_1(t)|t|u(m) \tag{3.5}$$

Proof. Applying (3) (since x_j and $S^{(1)}_{j,n}$ are associated by virtue of the property (P_4) from Esary, Proschan and Walkeep (1967)) and the representation $\exp(itx) = \cos(tx) + i\sin(tx)$ we get the estimate

$$\sum_{j\in A_n} Ex_j \exp\{\text{it } S^{(1)}_{j,n}\} = \sum_{j\in A_n} Ex_j(\cos(tS^{(1)}_{j,n}) + i\sin(tS^{(1)}_{j,n})) \le$$

$$\le 2|t|(\frac{L}{|A_n|^{\frac{1}{2}\sigma_0}})^2|A_n|u(m)$$

from this one can easily obtain (5) (note that A is bounded variable).

Now, let for $m, r \in N$

$$Q(m,1) = Q(m,2) = 0,$$

$$Q(m,r) = \max_{j\in A_n}\max_{1\le\ell\le r-2}\sum_{\rho=\ell+2}^{r} cov(U^{(\ell)}_{j,n}, U^{(s)}_{j,n}) \text{ as } r \ge 3.$$

and let $t \in R$ be such, that

$$\frac{|t|Lb_s(m,k+1)}{|A_n|^{\frac{1}{2}\sigma_0}} < \frac{1}{4}, \text{then the} \tag{3.6}$$

followings lemmas be proved:

Lemma 2 Let t satisfy (6), than

$$E|\zeta^{(1)}_j| \le (3m)^{\frac{d}{2}}\frac{L^2|t|c_s}{|A_n|\sigma_0^2}(u(0))^{\frac{1}{2}} := T_1(m,1) \tag{3.7}$$

Proof. Applying the Hölder inequality and $|e^{ix} - 1| \le x$ we get the estimate

$$E|\zeta^{(1)}_j| = E|x_j\xi^{(1)}_j| \le \frac{L^2}{|A_n|\sigma_0^2}|t|c_s(E(\sum_{k\in U^{(1)}_{j,n}}\epsilon_k)^2)^{\frac{1}{2}} \le$$

$$\le \frac{L^2}{|A_n|\sigma_0^2}|t|c_s(|U^{(1)}_{j,n}|u(0))^{\frac{1}{2}} \le \frac{L^2|t|}{|A_n|\sigma_0^2}(2m+1)^{\frac{d}{2}}u(0)^{\frac{1}{2}}.$$

Lemma 3

$$T_1(m,r) := E|\zeta_j^{(r)}| \le \frac{Lc_s}{|A_n|^{\frac{1}{2}}}((\Pi_{\ell=1}^r(E|\xi_j^{(e)}|^s)^{\frac{1}{s}} + r^{\frac{2}{s}}2^r(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0})^{\frac{4}{s}}Q(m,r)^{\frac{2}{s}} +$$

$$8r^{\frac{1}{s}}2^{\frac{r}{2}}c_s(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0})^{\frac{2}{s}}Q(m,r)^{\frac{1}{s}}) + 8u(m)^{\frac{s-2}{s-1}}((\frac{Lcs}{|A_n|^{\frac{1}{2}}\sigma_0})^{s-1} +$$

$$+ |t|^{\frac{s-1}{s}}(\frac{L}{|A_n|^{\frac{1}{2}}\sigma_0})^{\frac{3(s-1)}{s}}) \cdot (r^{\frac{1}{s}}2^r(\frac{|t|L}{|A_n|^{\frac{1}{2}}})^{\frac{2}{s}}Q(m,r)^{\frac{1}{s}} + 2^{-\frac{r}{2}}). \qquad (3.8)$$

Proof. In accordance with the Hölder inequality, we get an estimate

$$E|x_j\Pi_{\ell=1}^r\xi_j^{(\ell)}| \le (E\Pi_{\ell=1}^r{}'|\xi_j^{(\ell)}|^s)^{\frac{1}{s}}(E|x_j\Pi_{\ell=1}^r\xi_j^{(e)}|^{\frac{s}{s-1}})^{\frac{s-1}{s}} \qquad (3.9)$$

where $\Pi_{\ell=1}^r{}'$ and $\Pi_{\ell=1}^r{}''$ denote the products involving odd and even $\ell \in \{1,...,r\}$ respectively. By property (P_4) from Esary, Proschan and Walkup (1967), the random variables $U_{j,n}^{(\ell)}$ are associated. We have $|\xi_j^{(\ell)}|^s = 2^{\frac{s}{2}}(1-\cos(tU_{j,n}^{(e)}))^{\frac{s}{2}}$ and $|\xi_j^{(\ell)}| \le 2$.

Denote by $v(r)$ and $p(r)$ the number of odd and even integers in the set $\{\ell,...,r\}$ respectively, then $v(r-2) = v(r)-1$, when $r \ge 3$. Using (3) we have estimates

$$E\Pi_{\ell=1}^r{}'|\xi_j^{(\ell)}|^s \le |E|\xi_j^{(1)}|^s\Pi_{\ell=3}^r{}'E|\xi_j^{(\ell)}|^s + cov(|\xi_j^{(1)}|^s, \Pi_{\ell=3}^r{}'|\xi_j^{(\ell)}|^s) \le$$

$$\le ... \le \Pi_{\ell=1}^r{}'E|\xi_j^{(\ell)}|^s + (v(r)-1)2^{v(r)s}(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0})^2Q(m,r)$$

In accordance with inequality $(x+y)^\alpha \le x^\alpha + y^\alpha$ for all $x, y \ge 0, \alpha \in [0,1]$ we get such an estimate

$$(E\Pi_{\ell=1}^r{}'|\xi_j^{(\ell)}|^s)^{\frac{1}{s}} \le \Pi_{\ell=1}^r(E|\xi_j^{(\ell)}|^s)^{\frac{1}{s}} + (v(r)-1)^{\frac{1}{s}}2^{v(r)}\times$$

$$\times(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0})^{\frac{2}{s}}Q(m,r)^{\frac{1}{s}} \qquad (3.10)$$

Estimate for $\Pi_{\ell=1}^r{}''$ can be proved analogously.

$$(E\Pi_{\ell=1}^r{}''|\xi_j^{(\ell)}|^s)^{\frac{1}{s}} \le \Pi_{\ell=1}^r{}''(E|\xi_j^{(\ell)}|^s)^{\frac{1}{s}} + (p(r)-1)^{\frac{1}{s}}2^{p(r)}\times(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0})^{\frac{2}{s}}Q(m,r)^{\frac{1}{s}} \quad (3.11)$$

Following Birkel (1988), we introduce the function $\psi(x)$. Let $\psi(x) : R \to R$ denote for any $N > 0$ an odd, differentiable, nondecreasing function in such a way that $\psi(x) = x$ as $x \in [0, \frac{N}{2}], \psi(x) = N$ as $x \geq N$ and $\psi'(x) \leq 1$ for all x. Then for $\gamma = s(s-2)/(s-1)$ and any $x \in R$ one obtains

$$|x|^{\frac{s}{s-1}} - |\psi(x)|^{\frac{s}{s-1}} \leq 2^{\gamma} N^{-\gamma} |x|^{s} \tag{3.12}$$

Consider now the following estimation:

$$(E|X_j \Pi_{\ell=1}^{r}{}'' \xi_j^{(e)}|^{\frac{s}{s-1}})^{\frac{s-1}{s}} \leq |cov(|X_j|^{\frac{s}{s-1}}, \Pi_{\ell=1}^{r}{}''|\xi_j^{(e)}|^{\frac{s}{s-1}})|^{\frac{s-1}{s}} +$$

$$(E|X_j|^{\frac{s}{s-1}})^{\frac{s-1}{s}} (E\Pi_{\ell=1}^{r}{}''|\xi_j^{(e)}|^{\frac{s}{s-1}})^{\frac{s-1}{s}} \leq$$

$$\leq |cov(|x_j|^{\frac{s}{s-1}} - |\psi(x_j)|^{\frac{s}{s-1}}, \Pi_{\ell=1}^{r}{}''|\xi_j^{(e)}|^{\frac{s}{s-1}})|^{\frac{s-1}{s}} +$$

$$+|cov(|\psi(x_j)|^{\frac{s}{s-1}}, \Pi_{\ell=1}^{r}{}''|\xi_j^{(e)}|^{\frac{s}{s-1}})|^{\frac{s-1}{s}} +$$

$$+ (E|x_j|^{\frac{s}{s-1}})^{\frac{s-1}{s}} (E\Pi_{\ell=1}^{r}{}''|\xi_j^{(e)}|^{\frac{s}{s-1}})^{\frac{s-1}{s}} \tag{3.13}$$

Using inequalities $|\xi_j^{(e)}| \leq 2$ and (11) we get the following estimate for the first summand in (13):

$$|cov(|x_j|^{\frac{s}{s-1}} - |\psi(x_j)|^{\frac{s}{s-1}}, \Pi_{\ell=1}^{r}{}''|\xi_j^{(e)}|^{\frac{s}{s-1}})|^{\frac{s-1}{s}} \leq$$

$$\leq 2^{s-1} N^{-(s-2)} 2^{p(r)} C_s^{s-1} (\frac{L}{|A_n|^{\frac{1}{2}\sigma_0}})^{s-1} \tag{3.14}$$

Let $h(x) = |\xi_j^{(e)}|^{\frac{s}{s-1}} = 2^{\frac{s}{2(s-1)}} (1 - \cos(tU_j^{(e)}))^{\frac{s}{2(s-1)}}$

Then $||h'(x)||_{\infty} \leq \frac{s}{2(s-1)} 2^{\frac{s}{s-1}} \frac{|t|L}{|A_n|^{\frac{1}{2}\sigma_0}}$

and $||(|\psi(x)|^{\frac{s}{s-1}})'||_{\infty} \leq \frac{s}{s-1} N^{\frac{1}{s-1}}$

Applying the lemma 3.1 form Birkel (1988) we get the estimate of second summand in (13)

$$\begin{aligned} &|cov(|\psi(x_j)|^{\frac{s}{s-1}}, \Pi_{\ell=1}^{r}{}''|\xi_j^{(e)}|^{\frac{s}{s-1}})|^{\frac{s-1}{s}} \leq \\ &\leq 2N^{\frac{1}{s}} 2^{\frac{r}{2}} (\frac{|t|L^3}{|A_n|^{\frac{3}{2}\sigma_0^3}})^{\frac{s-1}{s}} u(m)^{\frac{s-1}{s}} \end{aligned} \tag{3.15}$$

Let $m = [\beta \ln |A_n|]$, where $\beta > 0, |A| > 1$, and $[\cdot]-$ be the integer part of a number.

We will assume $u(m) > 0$ for the above m, since otherwise the field $\{x_j, j \in A_n\}$ is $m-$ dependent and the estimate (1) under consideration is valid (see Koval (1992)). Let $N = u(m)^{-\frac{1}{s-1}}$. Then we deduce from (12), (13), (14) and the Lyapunov inequality:

$$(E|x_j \Pi''^{r}_{\ell=1} \xi_j^{(e)}|^{\frac{s}{s-1}})^{\frac{s-1}{s}} \leq c_s \frac{L}{|A_n|^{\frac{1}{2}\sigma_0}} (\Pi''^{r}_{\ell=1} |\xi_j^{(e)}|^s)^{\frac{1}{s}} +$$

$$+2^{p(r)+3}(c_s^{s-1}(\frac{L}{|A_n|^{\frac{1}{2}\sigma_0}})^{s-1} + |t|^{\frac{s-1}{s}}(\frac{L}{|A_n|^{\frac{1}{2}\sigma_0}})^{\frac{3(s-1)}{s}})u(m)^{\frac{s-2}{s-1}}$$

Thus, for t satisfying (6) and $r = 2, ..., k+1$ one has:

$$E|\xi_j^{(r)}| \leq \frac{L}{|A_n|^1/2\sigma_0} c_s (\Pi^r_{\ell=1}(E|\xi_j^{(e)}|^s)^{\frac{1}{s}} + r^{\frac{2}{s}} 2^r (\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0})^{\frac{4}{s}} \times$$

$$\times Q(m,r)^{\frac{2}{s}} + r^{\frac{1}{2}} 2^{v(r)+1} (\frac{|t|L}{|A_n|^{\frac{1}{2}}})^{\frac{2}{s}} Q(m,r)^{\frac{1}{s}} \times$$

$$\times (\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0} b_s(m,r))^{p(r)})+$$

$$+2^{p(r)+2}(c_s^{s-1}(\frac{L}{|A_n|^1/2\sigma_0})^{s-1}|t|^{\frac{s-1}{s}}(\frac{L}{|A_n|^1/2\sigma_0})^{\frac{3(s-1)}{s}} u(m)^{\frac{s-2}{s-1}} \times$$

$$\times \{(\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0} b_s(m,r))^{p(r)} + 2^{w(r)} r^{\frac{1}{s}} (\frac{|t|L}{|A_n|^{\frac{1}{2}}\sigma_0})^{\frac{2}{s}} Q(m,r)^{\frac{1}{s}}\}$$

Remembering that $(E|\xi_j^{(e)}|^s)^{\frac{1}{s}} \leq$
$\leq \frac{|t|L}{|A_n|^1/2\sigma_0}(E|U_{j,n}^{(e)}|^s)^{\frac{1}{s}}$, we get the assertion of Lemma 3.

Lemma 4 For all $t \in R$

$$i \sum_{j \in A_n} E\xi_j^{(1)} = -t + t|1 - \frac{\sigma_n^2}{d_n^2 \sigma_0^2}| + t\theta_2(t) c_1 u(m) +$$

$$+ 2\theta_3(t) c_s (\frac{|t|L b_s(m,1)}{|A_n|^{\frac{1}{2}}\sigma_0})^{s-1} \frac{c_2}{|A_n|^{\frac{1}{2}}} \qquad (3.16)$$

where c_i -are bounded variables.

Proof. Consider the representation $e^{ix} - 1 - ix = \theta(x)|x|^{1+\delta}$. Then

$$i\sum_{j\in A_n} E\xi_j^{(1)} = i\sum_{j\in A_n} Ex_j(it\sum_{k\in A^{(1)}_{j,n}} x_k+$$

$$+ 2\theta(t)|t\sum_{k\in V^{(1)}j,n} x_k|^{s-1}| \qquad (3.17)$$

The first summand estimates by

$$-\sum_{j\in A_n} Etx_j \sum_{k\in A_n} x_k = -t + t|1 - \frac{\sigma_n^2}{d_n^2\sigma_0^2}| \qquad (3.18)$$

For the second summand we obtain:

$$\sum_{j\in A_n} x_j t\sum_{k\in A^{(1)}_{j,n}} x_k = \theta_3(t)t|A_n|u(m)\frac{L^2}{|A_n|^1/2\sigma_0} = \theta_3(t)tu(m)\frac{L^2}{\sigma_0^2} \qquad (3.19)$$

In accordance with the Hölder inequality, we get finally for the last summand

$$\sum_{j\in A_n} x_n|t\sum_{k\in V^{(1)}_{j,n}} x_k|^{s-1} \leq (\frac{L}{|A_n|^1/2\sigma_0})^s|t|^{s-1}\sum_{j\in A_n}(E|\epsilon_j|^s)^{\frac{1}{s}}\times$$

$$\times((E(\sum_{k\in V^{(1)}_{j,n}} \epsilon_k)^{s-1})^{\frac{s-1}{s}} \leq$$

$$\leq (\frac{L}{|A_n|^1/2\sigma_0})^s|t|^{s-1}|A_n|c_s(b_s(m,1))^{s-1} \qquad (3.20)$$

The assertion of the lemma 4 follows from (16)-(19) with the following .....

$$\delta(m,r) := \max_{j\in A_n}\sum_{\ell=1}^{r}\sum_{s\geq r+2} cov(u^{(e)}_{j,n}, u^{(s)}_{j,n}), r\in N$$

the following lemmas proof the necessary estimates:

Lemma 5 For all $r\in N, j\in A_n$ and t satisfying (6) the following inequalities hold

$$|cov(\xi_j^{(n)}, \exp\{its^{(r+1)}_{j,n}\})| \leq \frac{L}{|A_n|^1/2\sigma_0}2^{2r+2}(\frac{|t|L}{|A_n|^1/2\sigma_0}u(m)+$$

$$+\delta(m,r)^{\frac{s-1}{s}}((\frac{|t|L}{|A_n|^1/2\sigma_0})^2 + c_s^s(\frac{L}{|A_n|^1/2})^s) := T_2(m,r) \qquad (3.21)$$

$$|\sum_{j\in A_n} E\xi_j^{(r)}(\exp\{its_j^{(r+1)}\} - f_n(t))| \leq |A_n|T_2(m,r)+$$

$$+5^{\frac{d}{2}}u(0)^{\frac{1}{2}}|A_n|T_1(m,r)(rm)^{\frac{d}{2}}\frac{|t|L}{|A_n|^1/2\sigma_0}|f_n(t)|+$$

$$+\ 2^{5d}|A_n|^{\frac{1}{2}}T_1(m,r)(rm)^d\frac{|t|L}{|A_n|^1/2\sigma_0}u(0)^{\frac{1}{2}} \tag{3.22}$$

The proofs of (20) and (21) are omitted since they are modifications of the proofs of Lemmas 3.3 and 3.4 in Birkel (1988) to the multidimensional case.

Lemma 6 For all $m, r \in N$, the following estimates are valid:

$$b_s(m,r) \leq d(3m)^d r^{d-1} c_s \frac{L}{|A_n|^1/2\sigma_0} \tag{3.23}$$

$$\delta(m,r) \leq d(3rm)^d \frac{L}{|A_n|^1/2\sigma_0}\sum\nolimits_{n\geq m} u(n) \tag{3.24}$$

$$Q(m,r) \leq d(3rm)^d \frac{L^2}{|A_n|^1/2\sigma_0}\sum\nolimits_{n\geq m} u(n) \tag{3.25}$$

Proof Since $|u_{j,n}^{(e)}| \leq 2md(2\ell m+1)^{d-1}$, then

$$b_s(m,r) = \max\nolimits_{j\in A_n}\max\nolimits_{1\leq \ell\leq r}(E|u_{j,n}^{(e)}|^s)^{\frac{1}{s}} \leq d(3m)^d r^{d-1}c_s \times \frac{L}{|A_n|^1/2\sigma_0}$$

¿From estimate $\delta(m,r) \leq u(r+1)m)+$
$+\sum\limits_{n=1}^{rm} d3^d n^{d-1}u((r+1)m+1-n)$ follows inequality (23). Estimate (24) follows from (23). Now, we estimate $\sum\limits_{r=2}^{k} T_1(m,r)$. For all t satisfying (6)

$$\sum\nolimits_{r=2}^{k} T_1(m,r) \leq c_s\frac{L}{|A_n|^1/2\sigma_0}(\frac{|t|L}{|A_n|^1/2\sigma_0}d(3m)^d2^{d-1}\frac{L}{|A_n|^1/2\sigma_0})^2\times$$

$$\times\sum\nolimits_{r=2}^{k}4^{-(r-2)} + c_s\frac{L}{|A_n|^1/2\sigma_0}Q(m,k)^{\frac{2}{s}}k^{\frac{2}{s}}2^{k+1}(\frac{|t|L}{|A_n|^1/2})^{\frac{4}{s}}\times$$

$$+4c_s\frac{L}{|A_n|^1/2\sigma_0}(Q(m,k))^{\frac{1}{s}}(\frac{|t|L}{|A_n|^1/2\sigma_0})^{\frac{2}{s}}\sum\nolimits_{r=2}^{\infty}r^{\frac{1}{s}}2^{-\frac{r}{2}}+8\{\sum\nolimits_{r=2}^{k}2^{-\frac{r}{2}}+$$

$$+k^{\frac{1}{s}}2^{k+1}Q(m,k)^{\frac{1}{s}}(\frac{|t|L}{|A_n|^1/2})^{\frac{2}{s}}\}\cdot((\frac{L}{|A_n|^1/2\sigma_0})^{s-1}c_s^{s-1}+$$

$$|t|^{\frac{s-1}{s}}(\frac{L}{|A_n|^1/2\sigma_0})^{\frac{3(s-1)}{s}})u(m)^{\frac{s-2}{s-1}} \leq$$

$\leq 2c_s t^2(\frac{L}{|A_n|^1/2\sigma_0})^5(b_s(m,2))^2 + c|A|^{-1}\frac{L}{|A_n|^1/2\sigma_0}G_s$
where $G_s = (\frac{L}{|A_n|^1/2\sigma_0})^2 c_s|t| + |t|^{\frac{s+1}{s}}(\frac{L}{|A_n|^1/2\sigma_0})^{\frac{3s-1}{s}} +$

$$+(c_s + c_s^{s-1})(\frac{|t|L}{|A_n|^1/2\sigma_0})^{\frac{2}{s}} + |t|^{\frac{s-1}{s}}(\frac{L}{|A_n|^1/2\sigma_0})^{\frac{3(s-1)}{s}} + c_s^{s-1}\frac{L}{|A_n|^1/2}$$

Let now $k = [\beta, \ln|A_n|], m[\beta_2 \ln|A_n|], \beta_1 > 2, \beta_2 > \beta_1$, choose $\beta_1 > \log_4 e$, to show the validity of the inequality $4^{-(k+1)} \leq |A_n|^{-1}$. Using condition 3) of the theorem, estimates (23),(24) can easily be obtained by:

$$\begin{aligned} k^{\frac{2}{s}}2^k Q(m,k+1)^{\frac{1}{s}} &\leq c_2|A_n|^{-1} \\ u(m)^{\frac{s-2}{s-1}} &\leq c_3|A_n|^{-1} \\ u(m)2^{2k} &\leq c_4|A_n|^{-1} \end{aligned}$$

$$2^{2k}((km)^{d-1}\sum\nolimits_{n\geq m} u(m))^{\frac{s-1}{s}} \leq c_5|A_n|^{-1}$$

where c_i, are variables, which depend on $d, s, u(x)$.

Hence

$$\sum\nolimits_{r=1}^{k} r^d T_1(m,r) \leq (3m)^{\frac{d}{2}} c_s \frac{|t|L^2}{|A_n|^1/2\sigma_0}((u(0))^{\frac{1}{2}} + 8\sum\nolimits_{r=2}^{\infty} r^d 4^{-r}(\frac{L}{|A_n|^1/2})^3 \times$$

$$\times (tb_s(m,2))^2 + c_2|A_n|^{-1}G_s \tag{3.26}$$

$$T_1(m,k+1) \leq c_3|A_n|^{-1}(c_s\frac{L}{|A_n|^1/2} + G_s \tag{3.27}$$

$$\sum\nolimits_{r=1}^{k} T_2(m,r) \leq c_4|A_n|^{-1}(\frac{|t|L}{|A_n|^1/2\sigma_0} + (\frac{tL}{|A_n|^1/2\sigma_0})^2 + (\frac{c_sL}{|A_n|^1/2\sigma_0})^s) \tag{3.28}$$

$$u(m) \leq c_s|A_n|^{-1} \tag{3.29}$$

Let t is such, that for β, and β_2 she has:

$$\frac{|t|L}{|A_n|^1/2\sigma_0} d6^d \beta_1^{d-1}\beta_2^d c_s(\ln|A_n|)^{2d-1} < \frac{1}{4}$$

then (6) is true and $b_s(m,2) \leq d6^d c_s m^d$.

Note that $G_s \leq a_0$, where $a_0 = a_0(d, c_s)$.

Based on lemmas 1-6 and estimates (25)-(28) and equation (2), may be rewritten in the following form

$$f_n'(t) = -tf_n(t) + t|1 - \frac{\sigma_n^2}{d_n^2\sigma_0^2}|f_n(t) + \theta_4(t)A(t)f_n(t) + \theta_s(t)B(t) \qquad (3.30)$$

where $A(t) = |A_n|^{-\frac{1}{2}}t^2(\ln|A_n|)^{2d} + |t|^{s-1}|A_n|^{-(\frac{s}{2}-1)}(\ln|A_n|)^{d(s-1)}$

$$B(t) = |A_n|^{-\frac{1}{2}} + |A_n|^{-\frac{1}{2}}t^2(\ln|A_n|)^{2d}$$

By integrating (29) for $|t| \le T_0$ and appling Esseen's inequality one can easily deduce from (29) (see Tikhomirov (1980)) that

$$\triangle_n \le c\{|A_n|^{-\frac{1}{2}}(\ln|A_n|)^{2d} + |A_n|^{-(\frac{s}{2}-1)}(\ln|A_n|)^{d(s-1)} + |1 - \frac{\sigma_n^2}{d_n^2\sigma_0^2}|\times$$

$$+|A_n|^{-\frac{1}{2}}\ln|A_n| + |A_n|^{-\frac{1}{2}}(\ln|A_n|)^{2d-1}\}$$

Then the estimate (1) is true, and the theorem is proved.

Acknowledgements: This paper was written while one of the authors was visiting Freie Universität Berlin in January-February 1998 under a grant from German Academic Exchange Service (DAAD).

References

1. Birkel, T. (1988) "On the convergence rate in the central limit theorem for associated processes".
   Ann. Probab. 16, No.4, 1685-1698

2. Bulinskii, A. (1995) "Rate of convergence in the central limit theorem for fields of associated random variables".
   Theory Probab. Appl. 40, No.1, 165-174. (in Russian)

3. Eberling, S. (1986) "Dependence in Probability and Statistics". A survey of resent results.
   Springer-Verlag, New York.

4. Esary, J., Proschan, F., Walkup, D. (1967) "Association of random variables with applications".
   Ann. Math. Statist. 38, No.5, 1466-1474.

5. Koval, T. (1992) "On the accuracy of normal approximation of regression coefficients of a random field".
   Theor. Probab. and Math. Statist. No. 47, 41-48 (in Russian)

6. Koval, T. (1996) "Central limit theorem for regression coefficients of associated random fields".
   Deponir. manuscr. Kiev (in Russian)

7. Newman, C. (1980) "Normal fluctuation and FKG inequalities".
   Commun. Math. Phys. 74, No.2, 119-128

8. Synklodas, I. (1986) "The estimate of the convergence rate in the central limit theorem for weakly dependent random fields".
   Lith. Math. J. 26, 541-559 (in Russian)

9. Tikhomirov (1980) "On the convergence rate in the central limit theorem for weakly dependent random variables".
   Theory Probab. Appl. 25, No.4, 790-808 (in Russian)

10. Wood, T. (1983) "A Berry-Esseen theorem for associated random variables".
    Ann. Probab. 11, No.4, 1042-1047