Traveling Waves on Lattice Dynamical Systems

A. A. Pankov and K. Pflüger

Preprint Nr. 3 A 99

Abstract. We consider traveling wave solutions for nonlinear lattices of particles with nearest neighbour interaction. By using variational methods we prove the existence of monotone traveling waves for finite, periodic lattices and show that in the limit, as the period goes to infinity, these waves converge to corresponding solutions of the infinite lattice.

AMS–Classification: 34K05, 34A45, 49K15, 70K99, 34C25

Keywords: nonlinear lattice, traveling waves, infinite dimensional Hamiltonian system, periodic lattice, variational methods.

A. A. Pankov: Dept. of Mathematics, Vinnitsa Pedagogical Institute, 287100 Vinnitsa, Ukraine. email: pankov@apmth.rts.vinnica.ua
K. Pflüger: Institut für Mathematik I, Freie Universität Berlin, Arnimalle 3, 14195 Berlin. email: pflueger@math.fu-berlin.de

1 Introduction

This paper concerns traveling waves for nonlinear lattices of particles with nearest neighbour interaction. Corresponding equations of motion are of the form

$$\ddot{q}_n = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1}), \quad n \in \mathbb{Z}, \tag{1}$$

where q_n denotes the coordinate of the nth particle and $V(r)$ is the potential of interaction. Equation (1) can be written as an infinite dimensional Hamiltonian system with Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} (\frac{1}{2}p_n^2 + V(q_{n+1} - q_n)),$$

where $p_n = \dot{q}_n$ is the momentum.

After the famous work by E. Fermi, J. Pasta and S. Ulam [9], such systems are of common interest due to their unexpected properties. First of all, the well-known Toda lattice defined by

$$V(r) = ab^{-1}(\exp(-br) + br - 1)$$

is a completely integrable system and its dynamics is understood very well [18]. There is also a number of other potentials, like the power potential, the Lennard-Jones potential, etc., which are of interest for many applications. However, until the beginning of the 90's dynamical properties of equation (1) with potentials of such kind were studied mainly by means of numerical simulations (see, e.g., [7, 8]).

In the 90's, due to the implementation of variational methods, the situation was changed and a number of rigorous results was obtained in the case of the general equation (1). In [1, 2, 3, 4, 5, 16] periodic motions for lattice systems were studied (even for time dependent and spatially inhomogeneous cases).

In the case of a general superquadratic potential, the first rigorous study of traveling waves, i.e. solutions of the form $q_n(t) = u(n - ct)$, was carried out in [11]. Making use of a constrained minimization approach, those authors obtained an existence result for solitary traveling waves with prescribed averaged potential energy. Then the wave speed c is determined in terms of the corresponding Lagrange multiplier.

A more direct approach to the existence problem for solitary waves has been suggested in a recent paper [17]. Here the Mountain Pass Theorem [20] was applied to prove the existence of solitary waves with prescribed wave speed c lying in a natural range.

We have also to point out a recent paper [10] containing, among other things, a number of qualitative properties of lattice solitary waves, mainly in the case of near sonic speed ($c - c_0 > 0$ is small enough, in our notations).

In the present paper we study both solitary and time periodic traveling waves. Our starting point is the following observation. For the Toda lattice, the explicit expression for k-periodic waves contains, as the limit case $k = \infty$, the Toda soliton, i.e. periodic waves give rise to solitary ones in the large wave length limit [18]. For general potentials we first prove the existence of k-periodic traveling waves, i.e. waves whose profile function $u(t)$ has k-periodic derivative. Of course, for corresponding lattice solution $\dot{q}_n(t) = -cu'(n - ct)$ is kc-periodic in time, but only almost periodic with respect to the spatial variable n in general. Such a solution is periodic in n only in the case of integer k. Then we examine the behavior of such waves in the large wave length limit, $k \to \infty$, and prove that (after translations and passage to a subsequence) they converge to a solitary traveling wave in an appropriate local topology. Remark that in the case of Lennard-Jones type potentials an existence result for periodic traveling waves was obtained in [19]. Under some additional, not so restrictive, assumptions we study so-called ground waves, i.e. waves with minimal possible averaged action. Here we use subsequently a constrained minimization approach known as the Nehari variational principle, or the Nehari manifold approach. We prove the existence of periodic ground waves and improve for such waves the above mentioned result on large wave length limit. Namely, in this case the limit is a solitary ground wave and convergence takes place on the entire period intervals, depending on k, i.e. in a sense globally. Roughly speaking, this means that a single bump of a periodic wave has almost the same shape as a solitary one whenever the wave length is large enough. Let us remark also that here we exploit the same ideas as in our previous papers [13, 14, 15].

Throughout this paper we consider only monotone waves (increasing, for definitness), i.e. waves with monotone profile function $u(t)$. However, similar results take place for nonmonotone waves as well (cf. [17]).

2 Variational framework

Solitary waves of equation (1), i. e. solutions of the form $q_n(t) = u(n-ct)$ are solutions of the equation

$$c^2 u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1))\,. \tag{2}$$

In addition to this infinite lattice, we also consider the corresponding problem on a periodic lattice consisting of $2k$ particles, denoted by $-k, -(k-1), \ldots, k$, where $-k$ and k are treated as the same point. This leads to the same equation as (1) (resp. (2)), but with an additional periodicity condition.

Equation (2) is equivalent to the following variational problem: Let X be the Hilbert space

$$X = \{u \in H^1_{loc} \mid u' \in L^2(\mathbb{R}),\ u(0) = 0\}$$

with norm $\|u\|^2 = \int_{\mathbb{R}} |u'(t)|^2 dt$. On X we define the operator A by

$$A(u)(t) = u(t+1) - u(t) = \int_t^{t+1} u'(s)\, ds\,.$$

Then a critical point of the functional

$$\Phi(u) = \int_{\mathbb{R}} [\frac{c^2}{2} u'(t)^2 - V(Au(t))]\, dt$$

on X is a solution of (1). The corresponding periodic problem is given by

$$\Phi_k(u) = \int_{-k}^{k} [\frac{c^2}{2} u'(t)^2 - V(Au(t))]\, dt,$$

on the Hilbert space of periodic functions

$$X_k = \{u \in H^1_{loc}(\mathbb{R}) \mid u'(t+2k) = u'(t),\ u(0) = 0\}$$

with norm $\|u\|_k^2 = \int_{-k}^{k} |u'(t)|^2 dt$.

Note that Φ and Φ_k are averaged Hamiltonian actions for u.

Monotone waves are fixed points of the operator $P : X_k \to X_k$, defined by $P(u)(t) = \int_0^t |u'(s)| ds$.

First we recall the following estimates for A which are proved in [17], Proposition 1:

Lemma 1 *A is a bounded operator from X to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (resp. from X_k to $L^2(-k,k) \cap L^\infty(-k,k)$) satisfying $\|Au\|_\infty \le \|u\|$, $\|Au\|_{L^2} \le \|u\|$ for all $u \in X$ (resp. $\|Au\|_\infty \le \|u\|_k$, $\|Au\|_{L^2} \le \|u\|_k$ for all $u \in X_k$).* □

In the sequel, we assume that the potential V satisfies the following:

(i) $V(r) = \frac{c_0^2}{2} r^2 + W(r)$, where $c > c_0 \geq 0$, $W \in C^1$, $W(0) = 0$, $W'(u) = o(|r|)$ as $r \to 0$.

(ii) $W(r_0) > 0$ for some $r_0 > 0$, and there exists some $\mu > 2$ such that $0 \leq \mu W(r) \leq W'(r)r$ for all $r \geq 0$.

Since we only consider monotone waves (i. e. $Au(t) \geq 0$), we may also assume that $W(r) = 0$ if $r \leq 0$, or we may modify W to be symmetric around 0. We note that Assumption (ii) can be written as a differential inequality

$$r^{\mu+1} \frac{d}{dr}(r^{-\mu} W(r)) \geq 0, \quad r > 0\,.$$

Integration of this inequality shows $W(r) \geq a_0 r^\mu$ for $r > r_0$ and $a_0 = r_0^{-\mu} W(r_0)$. Together with Assumption (i) this implies

$$W(r) \geq a_1(r^\mu - r^2) \quad \text{for } r \geq 0 \tag{3}$$

with some constant $a_1 > 0$. It follows from these assumptions and from Lemma 1, that Φ and Φ_k are differentiable; the derivative of Φ_k is given by

$$\langle \Phi_k'(u), v \rangle = \int_{-k}^{k} [c^2 u'(t)v'(t) - V'(Au(t))Av(t)]\, dt.$$

3 Periodic lattices

In this section we prove the existence of a sequence of periodic traveling waves by showing that Φ_k satisfies the assumptions of the Mountain Pass Theorem. Moreover, we derive several a priori estimates which enable us to pass to the limit $k \to \infty$ in the next section.

Lemma 2 *There exist $\delta, \rho > 0$ independent of k such that $|\Phi_k(u)| \geq \delta$ if $\|u\|_k = \rho$. Furthermore, there are functions $e_k \in P(X_k)$ satisfying $\Phi_k(e_k) = \Phi_1(e_1) \leq 0$ and $\|e_k\|_k > \rho$.*

Proof: Assumption (i) implies that, given $\varepsilon > 0$, there exists $\rho > 0$ such that $|W(r)| \leq \varepsilon |r|^2$ if $|r| \leq \rho$. Together with Lemma 1 this implies

$$\begin{aligned} \Phi_k(u) &\geq \int_{-k}^{k} [\frac{c^2}{2}|u'|^2 - \frac{c_0^2}{2}|Au|^2 - \varepsilon |Au|^2] dt \\ &\geq \frac{c^2 - c_0^2 - 2\varepsilon}{2} \|u\|_k^2, \end{aligned}$$

and ε, ρ can be choosen such that the right hand side is stricly positive for $\|u\|_k = \rho$. To construct the functions e_k, we first choose a function $v \in P(X_1)$ satisfying

$$v(t) = 0 \ \text{ for } \ 0 \le t \le 1, \quad v'(-1) = 0$$

(v' is nonzero only on intervals $[2l-1, 2l]$, $l \in \mathbb{Z}$) and $V(Av(t_0)) > 0$ for some t_0. Since $Av \ge 0$, we claim from (3) that

$$\begin{aligned} \Phi_1(\tau v) &\le \tau^2 \int_{-1}^{1} [\frac{c^2}{2}|v'|^2 + (a_1 - \frac{c_0^2}{2})|Av|^2]dt - \tau^\mu a_1 \int_{-1}^{1} |Av|^\mu dt \\ &\le \tau^2 \left(\frac{c^2}{2}\|v'\|_1^2 + a_1 \|Av\|_{L^2}^2 \right) - \tau^\mu a_1 \|Av\|_{L^\mu}^\mu. \end{aligned}$$

This shows $\Phi_1(\tau v) \to -\infty$ as $\tau \to \infty$, so we can fix $e_1 = \tau_0 v$ satisfying $\Phi_1(e_1) \le 0$ and $\|e_1\|_k > \rho$. Then we define $e_k \in X_k$ by

$$e_k(t) = e_1(t) \quad \text{for } \ |t| \le 1, \quad e_k'(t) = 0 \quad \text{for } \ 1 \le |t| \le k$$

and extend e_k' periodically to $\mathbb{R}$ (then e_k' is nonzero only on intervals $[2kl-1, 2kl], l \in \mathbb{Z}$).

It follows that $\|e_k\|_k = \|e_1\|_1$, and we have

$$e_k(t+1) - e_k(t) = \begin{cases} e_1(t+1) - e_1(t) & , t \in [-2, 0] \\ 0 & , t \in [-k, k] \setminus [-2, 0] \end{cases} .$$

Consequently

$$\begin{aligned} \int_{-k}^{k} V(Ae_k(t))dt &= \int_{-2}^{-1} V(e_1(t+1) - e_1(t))dt + \int_{-1}^{0} V(e_1(t+1) - e_1(t))dt \\ &= \int_{0}^{1} V(e_1(t-1) - e_1(t-2))dt + \int_{-1}^{0} V(e_1(t+1) - e_1(t))dt \\ &= \int_{-1}^{1} V(e_1(t+1) - e_1(t))dt, \end{aligned}$$

since the difference $e_1(t+1) - e_1(t)$ is 2-periodic. In particular, we have

$$\Phi_k(e_k) = \Phi_1(e_1) \le 0 \quad \text{ and } \quad \Phi_k(se_k) = \Phi_1(se_1) \quad \text{for } \ s \in \mathbb{R}. \tag{4}$$

□

Now we can prove the following theorem:

Theorem 1 *There exists a sequence $u_k \in P(X_k)$ of nontrivial critical points of Φ_k such that the critical values $c_k = \Phi_k(u_k)$ satisfy*

$$0 < \delta \le c_k \le M \tag{5}$$

with constants δ and M independent of k.

Proof: Since the embedding $X_k \to L^p(-k,k)$ is compact, we claim from Lemma 1 that Φ_k satisfies the Palais–Smale condition. Lemma 2 shows that Φ_k has the mountain pass geometry and $\Phi_k(P_k(u)) \le \Phi_k(u)$, $P_k(0) = 0$, $P_k(e_k) = e_k$. Consequently, the Brezis–Nirenberg version of the Mountain Pass Lemma (cf. [6], Thm. 10) shows that

$$\begin{aligned} c_k &= \inf_{\gamma\in\Gamma_k} \sup_{s\in[0,1]} \Phi_k(\gamma(s)), \quad \text{with} \\ \Gamma_k &= \{\gamma \in C([0,1], X_k) \mid \gamma(0) = 0, \gamma(1) \le 0, \ \|\gamma(1)\|_k > \rho\} \end{aligned}$$

are critical values of Φ_k with corresponding critical points $u_k \in \overline{P(X_k)}$. Since $P(X_k)$ is closed, $u_k \in P(X_k)$. Moreover, from (4) we claim

$$c_k \le \max_{s\in[0,1]} \Phi_k(se_k) = \max_{s\in[0,1]} \Phi_1(se_1) = M$$

and M is independent of k. □

4 Passage to the limit

To pass to the limit, we need a cut-off operator $T_k : X_k \to X$. which is defined as follows:

$$T_k u(t) = \begin{cases} u(t) & , |t| \le k+1 \\ u(\pm(k+1)) & , |t| \ge k+1 \end{cases}, \qquad u \in X_k.$$

Obviously, we have $(T_k u)' \in L^2(\mathbb{R})$ and

$$\|T_k u\|_X^2 = \int_{-\infty}^{\infty} |(T_k u)'|^2 dt = \int_{-k-1}^{k+1} |u'|^2 dt \le 2\|u\|_k^2,$$

so T_k is a bounded linear operator with norm independent of k. Remark that

$$(T_k A u)(t) = (A T_k u)(t) = Au(t) \quad \text{if} \quad |t| \le k, \tag{6}$$

and for increasing waves we have

$$0 \le (AT_k(u))(t) \le A(u(t)) \quad \text{for all} \quad t. \tag{7}$$

Lemma 3 *Any nontrivial critical point u of Φ_k satisfies $0 < \varepsilon_1 \le \|u\|_k \le C_k$, and $0 < \varepsilon_2 \le \Phi_k(u)$ with ε_1, ε_2 independent of k and C_k only depending on the critical value.*
The same assertions hold for critical points $u \in X$ of Φ.

Proof. To show the upper bounds for $\|u\|_k$, note that

$$\begin{aligned} c_k &= \Phi_k(u) - \frac{1}{2}\langle \Phi_k'(u), u\rangle \\ &= \int_{-k}^{k} [\frac{1}{2}W'(Au)Au - W(Au)]dt \\ &\ge \frac{\mu-2}{2}\int_{-k}^{k} W(Au)dt, \end{aligned}$$

by Assumption (ii). Therefore

$$\begin{aligned} \frac{c^2-c_0^2}{2}\|u\|_k^2 &\le \Phi_k(u) + \int_{-k}^{k} W(Au)dt \\ &\le \frac{\mu}{\mu-2}\, c_k. \end{aligned}$$

For the lower bounds, we assume on the contrary that there exists a sequence of nontrivial critical points $u_{k_n} \in X_{k_n}$ such that $\|u_{k_n}\|_{k_n} \le 1/n$. Since $\|Au_{k_n}\|_\infty \le 1/n$ (by Lemma 1), Assumption (i) shows $|W'(Au_{k_n})Au_{k_n}| \le \varepsilon_n |Au_{k_n}|^2$, $\varepsilon_n \to 0$. Hence

$$\begin{aligned} c^2\|u_{k_n}\|_{k_n}^2 &= \int_{-k_n}^{k_n} [c_0^2|Au_{k_n}|^2 + W'(Au_{k_n})Au_{k_n}]dt \\ &\le (c_0^2+\varepsilon_n)\,\|u_{k_n}\|_{k_n}^2 . \end{aligned}$$

Since $c^2 > c_0^2$, this is a contradiction.

Finally we obtain the lower bounds for $\Phi_k(u)$ as follows. First, from Assumption (ii) we have

$$\Phi_k(u) \ge \frac{c^2}{2}\|u\|_k^2 - \frac{c_0^2}{2}\|Au\|_{L^2}^2 - \frac{1}{\mu}\int_{-k}^{k} W'(Au)Au\,dt\,.$$

Then, by Lemma 1, we get

$$\begin{aligned} \Phi_k(u) &= \Phi_k(u) - \frac{1}{\mu}\langle \Phi_k'(u), u\rangle \\ &\ge (c^2-c_0^2)(\frac{1}{2}-\frac{1}{\mu})\,\|u\|_k^2 \\ &\ge (c^2-c_0^2)(\frac{1}{2}-\frac{1}{\mu})\,\varepsilon_1\,. \end{aligned}$$

Obviously, these lower bounds are independent of k, since they only depend on the geometry of the functional. Moreover, all the calculations above can be repeated for critical points $u \in X$ of Φ. □

Remark. Actually in the proof we have only used the fact that $\langle \Phi_k'(u), u \rangle = 0$, so the statements of this Lemma remain true under this weaker assumption.

For $\zeta \in \mathbb{R}$, $r > 0$ we denote by $K_r(\zeta)$ the interval with center ζ and length r.

Lemma 4 *Let $u_k \in X_k$ be a sequence such that $\Phi_k'(u_k) \to 0$ and $\|u_k\|_k \leq C$. Then either $\|u_k\|_k \to 0$, or there exist $r, \eta > 0$ and a sequence $\zeta_k \in \mathbb{R}$ such that, along a subsequence,*

$$\int_{K_r(\zeta_k)} |A(T_k u_k)|^2 dt \geq \eta.$$

Proof. First note that Lemma 1 implies that $A(T_k u_k)$ is a bounded sequence in $H^1(\mathbb{R})$. Now assume that

$$\lim_{k \to \infty} \sup_{\zeta \in \mathbb{R}} \int_{K_r(\zeta)} |A(T_k u_k)|^2 dt = 0.$$

Then the concentration lemma of Lions ([12], Lemma 1.1) shows that $\|A(T_k u_k)\|_{L^s(\mathbb{R})} \to 0$ for all $s > 2$. Since $\Phi_k'(u_k) \to 0$, there is a sequence $\varepsilon_k \to 0$ such that

$$\int_{-k}^{k} [c^2 (u_k')^2 - c_0^2 |Au_k|^2 - W'(Au_k) Au_k] \, dt = \varepsilon_k \|u_k\|_k \, .$$

Again Lemma 1 shows $\|Au_k\|_\infty \leq C$ and from Assumption (ii) we claim that for $|r| \leq C$, $\varepsilon > 0$ there exist $C_\varepsilon > 0$, $\alpha > 2$ such that $|W'(r)r| \leq \varepsilon r^2 + C_\varepsilon |r|^\alpha$. We get

$$\begin{aligned} c^2 \|u_k\|_k^2 &\leq \int_{-k}^{k} [(c_0^2 + \varepsilon)|Au_k|^2 + |Au_k|^\alpha] dt + \varepsilon_k \|u_k\|_k \\ &= (c_0^2 + \varepsilon) \|Au_k\|_{L^2}^2 + \int_{-k}^{k} |A(T_k u_k)|^\alpha dt + \varepsilon_k \|u_k\|_k \\ &\leq (c_0^2 + \varepsilon) \|u_k\|_k^2 + \|A(T_k u_k)\|_{L^\alpha(\mathbb{R})}^\alpha + \varepsilon_k \|u_k\|_k \, . \end{aligned}$$

Choosing ε small enough, this shows $\|u_k\|_k \to 0$ as desired. □

Theorem 2 *Let $u_k \in P(X_k)$ be a sequence of periodic waves such that the values $c_k = \Phi_k(u_k)$ satisfy the estimate (5). Then there exists a sequence $\zeta_k \in \mathbb{R}$ such that a subsequence of $T_k(u_k(\cdot+\zeta_k)-u_k(\zeta_k))$ converges weakly and locally in X to a nontrivial solution $u \in P(X)$ of (2).*

Proof. From (5) and Lemma 3 we conclude that $\|u_k\|_k$ is bounded both from above and below. Hence Lemma 4 and (7) imply that

$$\int_{K_r(\zeta_k)} |Au_k|^2 dt \geq \eta$$

for a subsequence. Now set $\tilde{u}_k(t) = u_k(t+\zeta_k) - u_k(\zeta_k)$. Then $\tilde{u}_k \in P(X_k)$, $\|\tilde{u}_k\|_k = \|u_k\|_k$, and since Φ_k is invariant under shifts and addition of constants, $\Phi_k(\tilde{u}_k) = \Phi_k(u_k)$, and $\Phi_k'(\tilde{u}_k) = 0$. Moreover, $T_k\tilde{u}_k$ converges weakly in X to some function u, and since $AT_k\tilde{u}_k$ is bounded in $H^1(\mathbb{R})$, we claim that $V'(AT_k\tilde{u}_k) \to V'(Au)$ strongly in $L^\infty_{loc}(\mathbb{R})$. Now for any function $\varphi \in C^\infty(\mathbb{R})$ with $\varphi(0) = 0$, $\varphi' \in C_0^\infty(\mathbb{R})$ (these functions are dense in X), we have

$$\begin{aligned}
\langle \Phi'(u), \varphi \rangle &= \lim_{k\to\infty} \int_{\mathbb{R}} [c^2 (T_k\tilde{u}_k)'\varphi' - V'(AT_k\tilde{u}_k)A\varphi]\, dt \\
&= \lim_{k\to\infty} \int_{-k}^{k} [c^2 \tilde{u}_k'\varphi' - V'(A\tilde{u}_k)A\varphi]\, dt \\
&= 0.
\end{aligned}$$

Hence u is a solution of (2) and clearly belongs to $P(X)$.

Since u_k is a solution of (2) with right hand side converging in L^∞_{loc}, we also have $u_k'' \to u''$ in L^∞_{loc}, and $u_k \to u$ locally in X. □

Theorem 3 *Let $u_k \in X_k$ be a uniformly bounded sequence satisfying $\Phi_k'(u_k) \to 0$ and $c_k = \Phi_k(u_k)u_k \to c > 0$. Assume that W is in C^2. Then there exist critical points $u^i \in X$ of Φ, $i = 1, \ldots, \ell$, and $\zeta_k^i \in \mathbb{R}$ such that*

$$\|u_k - \sum_{i=1}^{\ell} u^1(\cdot + \zeta_k^i)\|_k \to 0, \quad \sum_{i=1}^{\ell} \Phi(u_i) = c\,.$$

Proof. First note that the arguments in the proof of Theorem 2 only require that $\Phi_k'(u_k) \to 0$. Hence we claim as above that there exist $\zeta_k \in \mathbb{R}$ such that $\tilde{u}_k(t) = u_k(t+\zeta_k) - u_k(\zeta_k)$ satisfies $T_k\tilde{u}_k \rightharpoonup u$, a solution of (2).

Now let $\varphi_k \in C^\infty(\mathbb{R})$ be a sequence with the following properties: $\varphi_k(0) = 0$, $\varphi_k' \in C_0^\infty(\mathbb{R})$, $\mathrm{supp}(\varphi_k) \subset [-k+1, k-1]$, and $\varphi_k \to u$ strongly in X. Then $\varphi_k \in X \cap X_k$ and $A\varphi_k(t) = 0$ if $|t| \geq k$. Set $v_k = \tilde{u}_k - \varphi_k$. Since Φ is C^1, we get for an arbitrary $\eta_k \in X_k$:

$$\begin{aligned} \langle \Phi_k'(\varphi_k), \eta_k \rangle &= \langle \Phi'(\varphi_k), T_k\eta_k \rangle \\ &\leq \varepsilon_k \|T_k\eta_k\| \leq \varepsilon_k 2\|\eta_k\|_k. \end{aligned}$$

with $\varepsilon_k \to 0$, i. e. $\Phi'(\varphi_k) \to 0$.

Now we want to show that $\Phi_k'(v_k) \to 0$. First note that

$$\begin{aligned} \langle \Phi_k'(v_k), \eta_k \rangle &= \langle \Phi_k'(\tilde{u}_k) - \Phi_k'(\varphi_k), \eta_k \rangle \\ &\quad + \int_{-k}^{k} \left[W'(A\tilde{u}_k) - W'(A\varphi_k) - W'(A\tilde{u}_k - A\varphi_k) \right] A\eta_k dt. \end{aligned}$$

The convergence of $A\tilde{u}_k \to Au$ in L^∞_{loc} shows that for any $\varepsilon > 0$ and any bounded $B \subset \mathbb{R}$, we have

$$\left| \int_B \left[W'(A\tilde{u}_k) - W'(A\varphi_k) - W'(A\tilde{u}_k - A\varphi_k) \right] A\eta_k dt \right| \leq \varepsilon \|\eta_k\|_k,$$

if k is large enough. Denote $B^k = [-k, k] \setminus B$. From $\varphi_k \to u$ and $\Phi'(\varphi_k) \to 0$ we get $\|A\varphi_k\|_{L^\infty(B^k)} \leq \varepsilon$ and

$$\left| \int_{B_k} W'(A\varphi_k) A\eta_k dt \right| \leq \varepsilon \|\eta_k\|_k .$$

Since $W \in C^2$, we claim

$$\begin{aligned} \left| \int_{B_k} [W'(A\tilde{u}_k - A\varphi_k) - W'(A\varphi_k)] A\eta_k dt \right| &\leq \int_{B_k} |A\varphi_k| \, |1 + W''(A\tilde{u}_k)| \, |A\eta_k| dt \\ &\leq \|A\eta_k\|_{L^2} \left(\|A\varphi_k\|_{L^2(B^k)} + \|W''(A\tilde{u}_k)\|_{L^\infty(B^k)} \|A\varphi_k\|_{L^2(B^k)} \right) \\ &\leq \varepsilon \|\eta_k\|_k , \end{aligned}$$

hence $\Phi_k'(v_k) \to 0$.

Now consider

$$\begin{aligned} \Phi_k(\tilde{u}_k) &= \Phi_k(v_k) + \Phi_k(\varphi_k) + (v_k, \varphi_k)_k \\ &\quad - \int_{-k}^{k} \left[V(Av_k - A\varphi_k) - V(Av_k) - V(A\varphi_k) \right] dt. \end{aligned} \tag{8}$$

Since $v_k = \tilde{u}_k - \varphi_k$ tends locally to zero, we claim that $(v_k, \varphi_k)_k \to 0$ as $k \to \infty$. The integral in (8) can be estimated by similar arguments as above (using L^∞_{loc} estimates), so that we finally get

$$c = \lim_{k\to\infty} \Phi_k(\tilde{u}_k) = \Phi(u) + \lim_{k\to\infty} \Phi_k(v_k), \quad i.\,e.\ \Phi_k(v_k) \to c - J(u)\,. \tag{9}$$

Since $\Phi_k'(v_k) \to 0$, either $\Phi_k(v_k) \to 0$ and $\|v_k\|_k \to 0$, or $\Phi_k(v_k) \geq \varepsilon_2 > 0$ and $\|v_k\|_k \geq \varepsilon_1 > 0$ (cf. Remark following Lemma 3). In the first case we are done, while in the second case we have $0 < \varepsilon_2 \leq \Phi(u) < c$. Then we repeat the arguments above with u_k replaced by v_k and c replaced by $c - \Phi(u)$. □

5 Ground Waves

In this section we present another approach to construct travelling wave solutions of (1), which in addition have minimal energy among all nontrivial solutions. Here we use Nehari's variational principle and instead of Assumption (ii) we assume the following:

(N) $W \in C^2$, $W(r_0) > 0$ for some $r_0 > 0$, and there exists $\theta \in (0,1)$ such that $0 \leq r^{-1}W'(r) \leq \theta W''(r)$ holds for all $r \geq 0$.

As before, since we only consider monotone waves, we may assume that $W(r) \equiv 0$ for $r \leq 0$, or that W is symmetric around 0. We note that Assumption (N) implies (ii) with $\mu = (1+\theta)/\theta$. Now let

$$I_k(u) = \langle \Phi_k'(u), u\rangle = c^2 \|u\|_k^2 - \int_{-k}^{k} V'(Au)Au\,dt$$

and define the Nehari manifold by

$$S_k = \{u \in P(X_k) \setminus \{0\} \mid I_k(u) = 0\}\,.$$

Note that Assumption (N) implies that $W'(r)/r$ is monotone increasing for $r > 0$. Hence for any $u \in P(X_k) \setminus \{0\}$ there exists $\tau > 0$ such that $\tau u \in S_k$; moreover the number

$$\tau_u = \inf\{\tau > 0 \mid \tau u \in S_k\}$$

is unique.

Now we want to solve the periodic minimization problem

$$m_k = \inf\{\Phi_k(u) \mid u \in S_k\} \tag{10}$$

and the corresponding problem for the infinite lattice

$$m = \inf\{\Phi(u) \mid u \in S\} \tag{11}$$

(where $S \subset P(X)$ is defined in the same way as S_k above).
Note that on S_k the functional has the form

$$\Phi_k(u) = \int_{-k}^{k} [\frac{1}{2} W'(Au)Au - W(Au)]\, dt\,. \tag{12}$$

First we make the following useful observation:

Lemma 5 *Under Assumptions (i) and (N), $c_k = m_k$.*

Proof. Since c_k is a critical value with critical point $u_k \in P(X_k)$, u_k belongs to S_k, hence $m_k \leq c_k$.
If $v \in P(X_k)$, $v \not\equiv 0$, then as in the proof of Lemma 2 we see that $\Phi_k(\tau v) \leq 0$ for τ large enough. This shows $c_k \leq c_k'$, where c_k' is defined as

$$c_k' = \inf_{v \in P(X_k)\setminus\{0\}} \sup_{\tau > 0} \Phi_k(\tau v)\,.$$

Now observe that for $v \in P(X_k)$, $v \not\equiv 0$, there exists $\tau > 0$ such that $\Phi_k(\tau v) \geq \delta > 0$ (cf. Lemma 3). Since $\Phi_k(\tau v) \to -\infty$ as $\tau \to \infty$, the maximum of $\Phi_k(\tau v)$ is obtained for some $\tau_0 > 0$ and we have

$$\begin{aligned} 0 \;=\; \frac{d}{d\tau}\, \phi_k(\tau v)_{|\tau_0} \;&=\; \int_{-k}^{k} [c^2 \tau_0 |v'|^2 - V'(\tau_0 Av)Av]\, dt \\ &=\; \frac{1}{\tau_0}\, I_k(\tau_0 v)\,, \end{aligned}$$

hence $\tau_0 v \in S_k$. This shows $c_k' = \inf_{v \in S_k} \Phi_k(v) = m_k$. □

Since Assumption (N) implies (ii), from this Lemma and Theorem 1 we conclude the following:

Theorem 4 *Under Assumptions (i) and (N), for every k there exists a nontrivial minimizer u_k of (10) which is a solution of (2). Moreover, the sequence m_k is uniformly bounded from above and below.* □

Remark that any minimizer of (10) is a critical point of Φ_k (this follows from the Lagrange multiplier theorem and Assumption (N)). Moreover, the existence of minimizer can be proved directly by standard variational methods, since Φ_k has the necessary compactness properties.

For this ground waves we obtain the following stronger convergence result:

Theorem 5 *Under Assumptions (i) and (N), there exists a sequence $\zeta_k \in \mathbb{R}$ and a ground wave $u \in X$ such that, along a subsequence, the functions $\tilde{u}_k = u_k(\cdot + \zeta_k) - u_k(\zeta_k)$ satisfy*

$$\lim_{k\to\infty} \|\tilde{u}_k - u\|_k = 0\,.$$

Proof. The existence of a nontrivial limit which is a solution of (2) follows by the same arguments as in the proof of Theorem 2. Note that Lemma 5 and 3 imply the uniform boundedness of u_k.

To prove that the limit is actually a ground state, first note that for any $v \in S$, $\varepsilon > 0$, there exist $v_k \in S_k$ such that $\Phi_k(v_k) \le \Phi(v) + \varepsilon$ if k is sufficiently large. To show this, take a sequence $\varphi_k \in X_k$, $\varphi_k' \in C_0^\infty(-k+1, k-1)$, $\varphi_k \to v$ in X, and let $\tau_k > 0$ be the unique minimal number such that $\tau_k\varphi_k \in S_k$. From $\Phi(\varphi_k) \to \Phi(v)$, $I(\varphi_k) \to I(v)$, we claim $\tau_k \to 1$ and $v_k = \tau_k\varphi_k$ satisfies $\Phi_k(v_k) = \Phi(v_k) \to \Phi(v)$.

Since $m_k \le \Phi_k(v_k)$, this implies $\limsup_{k\to\infty} m_k \le \Phi(v) + \varepsilon$ for all $v \in S$, $\varepsilon > 0$, hence

$$\limsup_{k\to\infty} m_k \le m.$$

On the other hand consider the function $g(r) = \frac{1}{2}W'(r)r - W(r)$. Due to Assumption (N), $g(Au) \ge 0$. Since $g(A\tilde{u}_k) \to g(Au)$ locally in L^∞, and

$$m_k = \int_{-k}^{k} g(A\tilde{u}_k)dt \ge \int_B g(A\tilde{u}_k)dt$$

for any bounded interval B and k large enough, we claim

$$\liminf_{k\to\infty} m_k \ge m.$$

Thus, $m_k \to m = \Phi(u)$ as desired.

The last statement of Theorem 5 now follows directly from Theorem 3 (set $c = m$, $c_k = m_k$, and $\ell = 1$). □

Acknowledgement. The main part of this work was carried out during the visit of the first author to the Humboldt University, Berlin, as a guest professor under support of the Deutsche Forschungsgemeinschaft (September - December 1998).

References

[1] G. Arioli, J. Chabrowski, Periodic motions of a dynamical system consisting in an infinite lattice of particles. *Dyn. Syst. Appl.* **6** (1997), 387–395.

[2] G. Arioli, F. Gazzola, Existence and numerical approximation of periodic motions of an infinite lattice of particles. *Z. Angew. Math. Phys.* **46** (1995), 898–912.

[3] G. Arioli, F. Gazzola, Periodic motions of an infinite lattice of particles with nearest neigbor interaction. *Nonlin. Anal., TMA* **26** (1996), 1103–1114.

[4] G. Arioli, F. Gazzola, S. Terracini, Multibump periodic motions of an infinite lattice of particles. *Math. Z.* **223** (1996), 627–642.

[5] G. Arioli, A. Szulkin, Periodic motions of an infinite lattice of particles: the strongly indefinite case. *Prepr., Dip. Mat. Univ. Milano* (1997).

[6] H. Berestycki, I. Capuzzo-Dolcetta, and L. Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems. *NoDEA* **2** (1995), 553–572.

[7] D. B. Duncan, J. Wattis, Approximations of solitary waves on lattices using weak and variational formulations. *Chaos, Solitons and Fractals* **2** (1992), 505–518.

[8] J. C. Eilbeck, Numerical studies of solitons on lattices, in "Nonlinear Coherent Structures in Physics and Biology", *Lect. Notes Phys.* **393** (1991), 143–150, Springer, Berlin.

[9] E. Fermi, J. Pasta, S. Ulam, Studies of nonlinear problems. *Los Alamos Sci. Lab. Rept.* **LA-1940** (1955); Reprinted in *Lect. Appl. Math.* **15** (1974), 143–156.

[10] G. Friesecke, R. L. Pego, Solitary waves on FPU lattices, I. Preprint.

[11] G. Friesecke, J. Wattis, Existence theorem for solitary waves on lattices. *Commun. Math. Phys.* **161** (1994), 391–418.

[12] P.-L. Lions, The concentration-compactness method in the calculus of variations. The locally compact case. I, II. *Ann. Inst. H. Poincaré, Anal. Non Lin.* **1** (1984), 109–145, 223–283.

[13] A. Pankov, K. Pflüger, On a semilinear Schrödinger equation with periodic potential. *Nonlin. Anal., TMA* **33** (1998), 593–609.

[14] A. Pankov, K. Pflüger, Periodic and solitary traveling wave solutions for the generalized Kadomtsev-Petviashvili equation. *Math. Meth. Appl. Sci.*, to appear.

[15] A. Pankov, K. Pflüger, On ground traveling waves for the generalized Kadomtsev-Petviashvili equations. *Math. Phys., Anal. Geom.*, to appear.

[16] B. Ruf, P. N. Srikanth, On periodic motions of lattices of Toda type via critical point theory. *Arch. Ration. Mech. Anal.* **126** (1994), 369–385.

[17] D. Smets, M. Willem, Solitary waves with prescribed speed on infinite lattices. *J. Funct. Anal.* **149** (1997), 266–275.

[18] M. Toda, "Theory of nonlinear lattices". Springer, Berlin etc., 1989.

[19] T. P. Valkering, Periodic permanent waves in an anharmonic chain with nearest neighbour interaction. *J. Phys.* **A 11** (1978), 1885–1897.

[20] M. Willem, "Minimax methods." Birkhäuser, Boston etc., 1996.