CLOSURE OPERATIONS FOR DIGITAL TOPOLOGY

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ABSTRACT. We associate with α-ary relations, $\alpha > 1$ an ordinal, closure operations and study their properties, in particular the connectedness. Our interest is especially focussed on the closure operations associated with the terse relations introduced. For each natural number $n > 1$ we define an n-ary terse relation on the set of integers and investigate the associated closure operations. (For $n = 1$ we get the known Khalimsky topology). All investigations are done with respect to applications of the closure operations to digital topology.

0. Introduction

Digital topology is a theory that has arisen in the late 1960's for the study of geometric and topological properties of digital images. It proved to be a useful tool for solving some problems of computer graphics and image processing. Despite the name, the original methods of the theory were based on graph-theoretical rather than topological methods ([5], [6], [10], [11]). It was only in the late 1980's that a topological approach to digital topology was used for the first time in [3]. Since then, there were published a series of papers developping this approach (e.g. [4], [7], [8]). Most of them use the product of n copies of the so called *Khalimsky topology* on the set $\mathbb{Z}$ of integers for the study of n-dimensional digital images. This product topology decomposes the digital space $\mathbb{Z}^n$ into two classes of points (called the *pure* and *mixed* points respectively) that behave similarly with respect to connectedness. Therefore the topological approach to digital topology based on the Khalimsky topology is useful for investigations of binary, i.e. black-and-white, digital images. In this note we propose a more general approach. We introduce a sequence of closure operations on $\mathbb{Z}$ whose first member coincides with the Khalimsky topology. The sequence has increasing fineness of its members, i.e., it provides a sequence of product closure operations on $\mathbb{Z}^n$ having increasing number of classes of points that behave similarly with respect to connectedness. Thus, these pruduct closure operations could be useful when dealing with colour digital images. All closure operations studied are associated with α-ary relations ($\alpha > 1$ ordinals) and this fact is profitably used in our topological investigations.

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1. Preliminaries

Throughout the paper, ordinals are identified with the sets of their predecessors and cardinals with initial ordinals (so that the Axiom of Choice is assumed). The (strict) order on the class of ordinals is denoted by $<$, thus $\alpha < \beta$ is equivalent to $\alpha \in \beta$ whenever α, β are ordinals.

By a *closure operation* u on a set X we mean a map u: $\exp X \to \exp X$ fulfilling $u\emptyset = \emptyset$, $A \subseteq X \Rightarrow A \subseteq uA$, and $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$. Such closure operations were studied by E. Čech in [1] (who called them topologies). A pair (X, u), where X is a set and u is a closure operation on X, is called a *closure space*. Given a pair u, v of closure operations on a set X, we put $u \leq v$ if $uA \subseteq vA$ for each $A \subseteq X$. Clearly, $\leq$ is a partial order on the set of all closure operations on X. A closure operation u on a set X is called *additive* or *idempotent* if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ or $A \subseteq X \Rightarrow uuA = uA$ respectively. A closure operation u on a set X which is both additive and idempotent is called a *Kuratowski closure operation* or briefly a *topology* and the pair (X, u) is called a *topological space*. According to [12], given a cardinal $\alpha > 0$, a closure operation u on a set X and the closure space (X, u) are called an S_α*-closure operation* and an S_α*-space* respectively if the following condition (the so-called S_α*-axiom*) is satisfied:

$A \subseteq X \Rightarrow uA = \bigcup\{uB;\ B \subseteq A,\ \text{card}\, B < \alpha\}$.

Clearly, if (X, u) is an S_1-space, then $X = \emptyset$. In [2] S_2-closure operations and S_2-spaces are called quasi-discrete. Of course, any S_2-closure operation is additive, and any S_α-closure operation is an S_β-closure operation whenever $1 < \alpha < \beta$. Since any closure operation on a set X is obviously an S_α-closure operation for each cardinal α with $\alpha > \text{card}\, X$, there exists a least cardinal α such that u is an S_α-closure operation. Such a cardinal is then an important invariant of the closure operation u. Evidently, if $\alpha \leq \aleph_0$, then any additive S_α-closure operation is an S_2-closure operation.

Many concepts known for topological spaces can be naturally extended to closure spaces. Given a closure space (X, u), a subset $A \subseteq X$ is called *closed* if $uA = A$, and it is called *open* if $X - A$ is closed. A closure space (X, u) is said to be a *subspace* of a closure space (Y, v) if $uA = vA \cap X$ for each subset $A \subseteq X$. We will speak briefly about a subspace X of (Y, v). A closure space (X, u) is said to be *connected* if $\emptyset$ and X are the only subsets of X which are both closed and open. A subset $X \subseteq Y$ is connected in a closure space (Y, v) if the subspace X of (Y, v) is connected. A maximal connected subset of a closure space is called a *component* of this space. All the basic properties of connected sets and components in topological spaces (see e.g. [9]) are preserved also in closure spaces. A closure space (X, u) is said to be a T_0*-space* if for any points $x, y \in X$ from $x \in u\{y\}$ and $y \in u\{x\}$ it follows that $x = y$, and it is called a $T_{\frac{1}{2}}$*-space* if each singleteon subset of X is closed or open. Given closure spaces (X, u) and (Y, v), a map $\varphi : X \to Y$ is said to be a *continuous map* of (X, u) into (Y, v) if $f(uA) \subseteq vf(A)$ for each subset $A \subseteq X$. If, moreover, φ is a bijection and $\varphi^{-1} : Y \to X$ is a continuous map of (Y, v) into (X, u), then φ is called a *homeomorphism* of (X, u) onto (Y, v). We say that closure spaces (X, u) and (Y, v) (and the closure operations u and v) are *homeomorphic* if there exists a homeomorphism of (X, u) onto (Y, v).

Let X be a set and $\alpha > 1$ an ordinal. We denote by X^α the set of all maps of

α into X, i.e., the set of all sequences $(x_i|\ i < \alpha)$ of type α consisting of elements of the set X. As usual, for any ordinal m also a sequence $(x_i|\ m \leq i < m + \alpha)$ is considered to be a sequence of type α, namely the sequence $(x_{m+i}|\ i < \alpha)$. If a sequence $(x_i|\ i < \alpha)$ is said to be a set, we mean the set $\{x_i;\ i < \alpha\}$. Any subset $R \subseteq X^\alpha$ is called an *α-ary relation* on X and the pair (X, R) is called an *α-ary relational system*. An α-ary relation on a set X is said to be *reflexive* if it contains all constant sequences of type α consisting of elements of the set X. Given α-ary relational systems (X, R) and (Y, S), a map $\varphi : X \to Y$ is called a *homomorphism* of (X, u) into (Y, v) if the implication $(x_i|\ i < \alpha) \in R \Rightarrow (\varphi(x_i)|\ i < \alpha) \in S$ is valid. If, moreover, φ is a bijection and $\varphi^{-1} : Y \to X$ is a homomorphism of (Y, S) into (X, R), then φ is called an *isomorphism* of (X, R) into (Y, S). We say that α-ary relational systems (X, R) and (Y, S) (and the α-ary relations R and S) are isomorphic if there exists an isomorphism of (X, R) into (Y, S).

Let (X_j, R_j), $j \in J$, be a system of α-ary relational systems. By the *product* of the system we understand the α-ary relational system $\prod\limits_{j\in J}(X_j, R_j) = (\prod\limits_{j\in J} X_j, \prod\limits_{j\in J} R_j)$ where $\prod\limits_{j\in J} R_j$ is the α-ary relation on the cartesian product $\prod\limits_{j\in J} X_j$ given by $\prod\limits_{j\in J} R_j = \{(x_i|\ i < \alpha) \in (\prod\limits_{j\in J} X_j)^\alpha; (pr_j(x_i)|\ i < \alpha) \in R_j$ for each $j \in J\}$. If the set J is finite, say $J = \{1, ..., m\}$, we write $R_1 \times ... \times R_m$ and $(X_1, R_1) \times ... \times (X_m, R_m)$ instead of $\prod\limits_{j\in J} R_j$ and $\prod\limits_{j\in J}(X_j, R_j)$, respectively.

2. Closure operations associated with relations

Throughout this paragraph, α denotes an ordinal with $\alpha > 1$, and $\langle\alpha\rangle$ denotes the least of all cardinals which are greater or equal to α.

Let X be a set and R an α-ary relation on X. Then we define a map u_R: $\exp X \to \exp X$ as follows:

$u_R A = A \cup \{x \in X;$ there exist $(x_i|\ i < \alpha) \in R$ and $i_0, 0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in A$ for all $i < i_0\}$.

Clearly, u_R is a closure operation on X. Note that in the case of a reflexive relation R we have

$u_R A = \{x \in X;$ there exist $(x_i|\ i < \alpha) \in R$ and $i_0, 0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in A$ for all $i < i_0\}$.

Proposition 2.1. *For any α-ary relation R on a set X, (X, u_R) is an $S_{\langle\alpha\rangle}$-space.*

Proof. Let $A \subseteq X$ and $x \in u_R A$. If $x \in A$, then $x \in u_R\{x\} \subseteq \bigcup\{u_R B;\ B \subseteq A$ and card $B < \langle\alpha\rangle\}$. Suppose $x \notin A$. Then there exist $(x_i|\ i < \alpha) \in R$ and i_0, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in A$ for all $i < i_0$. Hence $\{x_i;\ i < i_0\} \subseteq A$ and card$\{x_i;\ i < i_0\} \leq$ card i_0. But card $i_0 < \alpha = \langle\alpha\rangle$ whenever α is a cardinal and card $i_0 \leq$ card $\alpha < \langle\alpha\rangle$ whenever α is not a cardinal. Since $x \in u_R\{x_i;\ i < i_0\}$, we have $x \in \bigcup\{u_R B;\ B \subseteq A$ and card $B < \langle\alpha\rangle\}$. Therefore the inclusion $u_R X \subseteq \bigcup\{u_R B;\ B \subseteq A$ and card $B < \langle\alpha\rangle\}$ is valid and, since the converse inclusion is evident, the proof is complete.

Thus, if α is a cardinal and R is an α-ary relation on a set X, then (X, u) is an S_α-space.

Lemma 2.2. *Let R be an α-ary relation on a set X and $(x_i|\ i<\alpha)\in R$ an element. Then the set $\{x_i;\ i<i_0\}$ is connected in (X,u_R) for each i_0, $0<i_0\leq\alpha$.*

Proof. For $i_0=1$ the statement is trivial. Let $i_0>1$ and suppose that $\{x_i;\ i<i_1\}$ is connected for each i_1, $0<i_1<i_0$. As $\{x_i;\ i<i_1\}\subseteq\{x_i;\ i\leq i_1\}\subseteq u_R\{x_i;\ i<i_1\}$, the set $\{x_i;\ i\leq i_1\}$ is connected for each i_1, $0<i_1<i_0$. Since $\bigcap\limits_{0<i_1<i_0}\{x_i;\ i\leq i_1\}\neq\emptyset$, $\bigcup\limits_{0<i_1<i_0}\{x_i;\ i\leq i_1\}$ is connected. But $\bigcup\limits_{0<i_1<i_0}\{x_i;\ i\leq i_1\}=\{x_i;\ i<i_0\}$ and the statement follows from the principle of transfinite induction.

Definition 2.3. Let R be an n-ary relation on a set X where $n>1$ is a natural number (i.e., a finite ordinal) and let k be a natural number with $1<k\leq n$. A sequence $(y_i|\ i<k)$ of points of X is called a *connected element* in (X,u_R) if there is a sequence $(x_i|\ i<n)\in R$ such that $y_i=x_i$ for all $i<k$ or $y_i=x_{k-1-i}$ for all $i<k$.

By Lemma 2.2, each connected element is a connected set.

Definition 2.4. Let $n>1$ be a natural number and let R be an n-ary relation on a set X. A finite nonempty sequence $C=(x_i|\ i<m)$ of points of X is called a *path* in (X,u_R) if there is a finite increasing sequence $(j_k|\ k<p)$ of natural numbers with $j_0=0$ and $j_{p-1}=m-1$ such that $j_k-j_{k-1}<n$ and $(x_j|\ j_{k-1}\leq j\leq j_k)$ is a connected element in (X,u_R) for each k, $0<k<p$.

Clearly, each path is a connected set and each connected element is a path. If $(x_i|\ i<m)$ is a path, then also its *inverse*, i.e., the sequence $(y_i|\ i<m)$ where $y_i=x_{m-1-i}$ for all $i<m$, is a path. Further, if $(x_i|\ i<m)$, $(y_i|\ i<p)$ are paths such that $x_{m-1}=y_0$, then also their *union*, i.e., the sequence $(z_i|\ i<m+p-1)$ where $z_i=x_i$ for all $i<n$ and $z_i=y_{i-m+1}$ for all i with $m\leq i<p$, is a path.

Theorem 2.5. *Let $n>1$ be a natural number, let R be an n-ary relation on a set X, and let $A\subseteq X$ be a subset. Then A is connected in (X,u_R) if and only if any two points of A can be connected by a path in (X,u_R) contained in A (i.e., whenever $x,y\in A$, there is a path $(x_i|\ i<m)$ in (X,u_R) such that $x_0=x$, $x_{m-1}=y$, and $x_i\in A$ for all $i<m$).*

Proof. If $A=\emptyset$, then the statement is trivial. Let $A\neq\emptyset$. In (X,u_R), if any two points of A can be connected by a path, then A is clearly connected. Conversely, let A be connected and suppose that there are points $x,y\in A$ which can not be connected by a path contained in A. Let B be the set of all points of A which can be connected with x by a path contained in A. Let $z\in u_RB$ be a point and assume that $z\notin B$. Then there are $(x_i|\ i<n)\in R$ and i_0, $0<i_0<n$, such that $z=x_{i_0}$ and $\{x_i;\ i<i_0\}\subseteq B$. Thus, x and x_0 can be connected by a path contained in A, and also x_0 and z can be connected by a path - the connected element $(x_i|\ i\leq i_0)$ - contained in A. It follows that x and z can be connected by a path contained in A, which is a contradiction. Therefore $z\in B$. Consequently, B is closed in the space (X,u_R), hence also in the subspace A of (X,u_R). Further, let $z\in u_R(A-B)\cap A$ be a point and assume that $z\in B$. Then $z\notin A-B$, thus there are $(x_i|\ i<n)\in R$ and i_0, $0<i_0<n$, such that $z=x_{i_0}$ and $\{x_i;\ i<i_0\}\subseteq A-B$. As we have assumed that $z\in B$, x can be connected with z by a path contained in A. Further, z can be

connected with x_0 by a path - the connected element $(x_{i_0-i}|\ i \le i_0)$ - contained in A. Consequently, x and x_0 can be connected by a path contained in A, which is a contradiction with $x_0 \notin B$. Thus $z \notin B$, i.e. $z \in A - B$, which means that $A - B$ is closed in the subspace A of (X, u_R). Hence A is the union of the nonempty disjoint sets B and $A - B$ closed in the subspace A of (x, u_R). But this is a contradiction because A is closed. Therefore any two points of A can be connected by a path contained in A.

Definition 2.6. An α-ary relation R on a set X is said to be *terse* provided that it is reflexive and fulfils the following condition:

If $(x_i|\ i < \alpha) \in R$, $(y_i|\ i < \alpha) \in R$, and there are ordinals $i_0, i_1 < \alpha$, $i_0 \ne i_1$, such that $x_0 = y_{i_0}$ and $x_1 = y_{i_1}$, then $(x_i|\ i < \alpha) = (y_i|\ i < \alpha)$.

Thus, any nonconstant sequence belonging to a terse relation is injective. Clearly, a binary relation is terse if and only if it is reflexive and antisymmetric.

Proposition 2.7. *If R is a terse α-ary relation on a set X, then (X, u_R) is a T_0-space.*

Proof. Let $x, y \in X$, $x \in u_R\{y\}$, $y \in u_R\{x\}$. Then there is a sequence $(x_i|\ i < \alpha) \in R$ with $x_0 = y$ and $x_1 = x$, and there is a sequence $(y_i|\ i < \alpha)$ with $y_0 = x$ and $y_1 = y$. As R is terse, $(x_i|\ i < \alpha) = (y_i|\ i < \alpha)$. Therefore $x = y$, which proves the statement.

Proposition 2.8. *Let R be a terse relation on a set X. Then R is a minimal element (with respect to the set inclusion) of the set of all reflexive α-ary relations S on X fulfilling $u_R = u_S$.*

Proof. Let S be a reflexive α-ary relation on X with $u_R = u_S$ and let $S \subseteq R$. Let $(x_i|\ i < \alpha) \in R$ be an arbitrary sequence. If $(x_i|\ i < \alpha)$ is constant, then clearly $(x_i|\ i < \alpha) \in S$. Let $(x_i|\ i < \alpha)$ be not constant. Then $x_0 \ne x_1$ and $x_1 \in u_R\{x_0\}$, hence $x_1 \in u_S\{x_0\}$. Therefore there are $(y_i|\ i < \alpha) \in S$ and i_0, $0 < i_0 < \alpha$, such that $x_1 = y_{i_0}$ and $y_i = x_0$ for all $i < i_0$. Since $(y_i|\ i < \alpha) \in R$ and R is terse, we have $(x_i|\ i < \alpha) = (y_i|\ i < \alpha) \in S$. Thus $S = R$ and the proof is complete.

Given an ordinal $\alpha > 1$, for any closure operation u on X we put

$R_u = \{(x_i|\ i < \alpha) \in X^\alpha;\ (x_i|\ i < \alpha)$ is either constant or injective with the property that for each j, $0 < j < \alpha$, and each proper subset $A_j \subset \{x_i;\ i < j\}$ there holds $x_j \in u\{x_i; i < j\} - uA_j\}$.

Clearly, R_u is a reflexive α-ary relation on X.

Theorem 2.9. *If R is a terse α-ary relation, then $R = R_{u_R}$.*

Proof. Let $(x_i|\ i < \alpha) \in R$. If $(x_i|\ i < \alpha)$ is constant, then evidently $(x_i|\ i < \alpha) \in R_{u_R}$. Let $(x_i|\ i < \alpha)$ be injective and let i_0, $0 < i_0 < \alpha$, be an arbitrary ordinal. Clearly, $x_{i_0} \in u_R\{x_i;\ i < i_0\}$. Let $A \subseteq \{x_i;\ i < i_0\}$ be an arbitrary subset and let $x_{i_0} \in u_R A$. Then there exist an injective sequence $(y_j|\ j < \alpha) \in R$ and an ordinal j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j;\ j < j_0\} \subseteq A$, hence $\{y_j;\ j \le j_0\} \subseteq \{x_i;\ i \le i_0\}$. Thus $\{y_0, y_1\} \subseteq \{x_i;\ i < \alpha\}$ and consequently $(x_i|\ i < \alpha) = (y_j|\ j < \alpha)$. As $(x_i|\ i < \alpha)$ is injective, there holds $i_0 = j_0$ and therefore $A = \{x_i;\ i < i_0\}$. We have shown that $x_{i_0} \notin u_R A$ whenever $A \subseteq \{x_i;\ i < i_0\}$ is

a proper subset. It follows that $(x_i|\ i<\alpha)\in R_{u_R}$ which results in the inclusion $R\subseteq R_{u_R}$.
Conversely, let $(x_i|\ i<\alpha)\in R_{u_R}$. If $(x_i|\ i<\alpha)$ is constant, then clearly $(x_i|\ i<\alpha)\in R$. Let $(x_i|\ i<\alpha)$ be injective. As $x_1\in u_R\{x_0\}$ and R is terse, there is a unique sequence $(y_i|\ i<\alpha)\in R$ with $x_0=y_0$ and $x_1=y_1$. Suppose that there is an ordinal i_0, $1<i_0<\alpha$, such that $x_i=y_i$ for all $i<i_0$. Since $x_{i_0}\in u_R\{x_i;\ i<i_0\}$, there are an injective sequence $(z_j|\ j<\alpha)\in R$ and an ordinal j_0, $0<j_0<\alpha$, such that $x_{i_0}=z_{j_0}$ and $\{z_j;\ j<j_0\}\subseteq\{x_i;\ i<i_0\}=\{y_i;\ i<i_0\}$. Then $x_{i_0}\in u_R\{z_j;\ j<j_0\}$, hence $\{z_j;\ j<j_0\}=\{x_i;\ i<i_0\}$. Thus $\{z_0,z_1\}\subseteq\{y_i;\ i<i_0\}$. Consequently, $(z_j|\ j<\alpha)=(y_i|\ i<\alpha)$. Admit that $j_0<i_0$. Then $x_{j_0}=y_{j_0}=z_{j_0}$, which is a contradiction because $x_{i_0}=z_{j_0}$ and $(x_i|\ i<\alpha)$ is injective. Hence $i_0\le j_0$. Admit that $i_0<j_0$. Then there is an ordinal $i_1<i_0$ such that $z_{i_0}=x_{i_1}$, which is a contradiction because $z_{i_1}=y_{i_1}=x_{i_1}$ and $(z_i|\ i<\alpha)$ is injective. Therefore $i_0=j_0$ and we have $x_{i_0}=z_{i_0}=y_{i_0}$. Now the principle of transfinite induction implies $(x_i|\ i<\alpha)=(y_i|\ i<\alpha)$. We have shown that $(x_i|\ i<\alpha)\in R$, which results in the inclusion $R_{u_R}\subseteq R$. The proof is complete.

Corollary 2.10. *For terse α-ary relations the correspondence $R\mapsto u_R$ is one-to-one.*

Definition 2.11. Let X be a set and $\alpha>1$ an ordinal. A closure operation u on X and the closure space (X,u) are called an S_α^**-closure operation* and an S_α^**-space* respectively if there is a terse α-ary relation R on X such that $u=u_R$ (in which case $R=R_u$).

Corollary 2.12. *The assignments $R\mapsto u_R$ and $u\mapsto R_u$ determine bijections inverse to each other between the set of all terse α-ary relations on a set X and the set of all S_α^*-closure operations u on X.*

Clearly, the product of a system of terse α-ary relational systems is a terse α-ary relational system. This fact, together with Corollary 2.10, enables us to define:

Definition 2.13. Let (X_j,u_j), $j\in J$, be a system of S_α^*-spaces. By the product of this system we understand the S_α^*-space $\prod\limits_{j\in J}(X_j,u_j)=(\prod\limits_{j\in J}X_j,u_R)$ where $R=\prod\limits_{j\in J}R_{u_j}$.

If the set J is finite, say $J=\{1,...,m\}$, then we write $(X_1,u_1)\times...\times(X_m,u_m)$ instead of $\prod\limits_{j\in J}(X_j,u_j)$.

3. S_n^*-closure operations on $\mathbb{Z}$

In this paragraph, n denotes a natural number with $n>1$, and ω denotes the least infinite ordinal. For each $k\in\mathbb{Z}$ we define an n-ary relation $R_{n,k}$ on $\mathbb{Z}$ as follows:
$R_{n,k}=\{(x_i|\ i<n)\in\mathbb{Z}^n;\ (x_i|\ i<n)$ is constant or there exists an even number $l\in\mathbb{Z}$ fulfilling either $x_i=k+l(n-1)+i$ for all $i<n$ or $x_i=k+l(n-1)-i$ for all $i<n\}$.
Thus, a nonconstant sequence $(x_i|\ i<n)\in\mathbb{Z}^n$ belongs to $R_{n,k}$ if and only if $(x_i|\ i<n)$ is an arithmetic sequence with the difference equal to 1 or -1 and with $x_0=k+l(n-1)$ where $l\in\mathbb{Z}$ is an even number. Note that each element of

$z \in \mathbb{Z}$ belongs to at liest one and at most two nonconstant sequences from $R_{n,k}$. It belongs to two (different) nonconstant sequences from $R_{n,k}$ if and only if there is $l \in \mathbb{Z}$ with $z = k + l(n-1)$ (in which case z is the first member of each of the two sequence if l is even, and z is the last member of each of the two sequences if l is odd). Of course, $R_{2,k}$ is a partial order on $\mathbb{Z}$ for each $k \in \mathbb{Z}$. On the following picture, the nonconstant sequences of $R_{n,k}$ are represented as arrows oriented from the first to the last members of the sequences:

$$\cdots \longleftarrow\!\!\!-\!\!\!-\!\!\!- \qquad -\!\!\!-\!\!\!-\!\!\!\longrightarrow \; \longleftarrow\!\!\!-\!\!\!-\!\!\!- \qquad -\!\!\!-\!\!\!-\!\!\!\longrightarrow \; \longleftarrow\!\!\!-\!\!\!-\!\!\!- \qquad -\!\!\!-\!\!\!-\!\!\!\longrightarrow \cdots$$

$$k-3(n-1) \quad k-2(n-1) \quad k-(n-1) \qquad k \qquad k+n-1 \quad k+2(n-1) \quad k+3(n-1)$$

It is evident that $R_{n,k}$ is terse for each $k \in \mathbb{Z}$. Thus, $u_{R_{n,k}}$ is an R_n^*-closure operation for each $k \in \mathbb{Z}$ and we will denote it briefly by $u_{n,k}$. Clearly, for any $k_1, k_2 \in \mathbb{Z}$, R_{n,k_1} and R_{n,k_2} are isomorphic, hence u_{n,k_1} and u_{n,k_2} are homeomorphic. It is obvious that $R_{n,k_1}^{-1} = R_{n,k_2}$ whenever $k_1, k_2 \in \mathbb{Z}$, $k_2 = k_1 + n - 1$. Thus, if $k_1, k_2 \in \mathbb{Z}$, $k_2 = k_1 + 2n - 2$, then $R_{n,k_1} = R_{n,k_2}$ and consequently $u_{n,k_1} = u_{n,k_2}$. It follows that there are precisely $2n-2$ different but homeomorphic closure operations among all $u_{n,k}$, $k \in \mathbb{Z}$. So it is sufficient to investigate only one of them. We will investigate the closure operation $u_{n,0}$ which will be denoted briefly by u_n. Analogously, the relation $R_{n,0}$ will be denoted by R_n. The closure operation u_2 coincides with the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k, 2k+1, 2k+2\};\ k \in \mathbb{Z}\}$ (see [3]). The relation $R_2^{-1} = R_{2,1}$ is the so-called specialization order of u_2. (The closure operation $u_{2,1}$ coincides with the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k-1, 2k, 2k+1\};\ k \in \mathbb{Z}\}$ and $R_{2,1}^{-1} = R_2$ is the specialization order of $u_{2,1}$.)

Proposition 3.1. *In the S_n^*-space $(\mathbb{Z}, u_n)$, the points $l(n-1)$, $l \in \mathbb{Z}$ even, are open, while all the other points are closed.*

Proof. Let $l_0 \in \mathbb{Z}$ be an arbitrary even number and put $z = l_0(n-1)$. Let $x \in u_n(\mathbb{Z} - \{z\})$ be a point. Then there are $(x_i|\ i < n) \in R_n$ and i_0, $0 < i_0 < n$, such that $x = x_{i_0}$ and $\{x_i;\ i < i_0\} \in \mathbb{Z} - \{z\}$. If $(x_i|\ i < n)$ is constant, then clearly $x \in \mathbb{Z} - \{z\}$. Let there exists an even number $m \in \mathbb{Z}$ fulfilling either $x_i = m(n-1)+i$ for all $i < n$ or $x_i = m(n-1) - i$ for all $i < n$. Then $x_{i_0} = m(n-1) + i_0$ or $x_{i_0} = m(n-1) - i_0$. Since $n \leq 2(n-1)$, we have $0 < i_0 < 2(n-1)$ and consequently $m(n-1) < x_{i_0} < (m+2)(n-1)$ or $(m-2)(n-1) < x_{i_0} < m(n-1)$. Hence $x_{i_0} \neq l(n-1)$ for any even number $l \in \mathbb{Z}$. Thus $x_{i_0} \neq z$, so that $x \in \mathbb{Z} - \{z\}$. Therefore the set $\mathbb{Z} - \{z\}$ is closed, i.e., $\{z\}$ is open.
Let $z \in \mathbb{Z}$ be an arbitrary point with $z \neq l(n-1)$ for any even number $l \in \mathbb{Z}$. Let $x \in u_n\{z\}$. Then there are $(x_i|\ i < n) \in R_n$ and i_0, $0 < i_0 < n$, such that $x = x_{i_0}$ and $\{x_i;\ i < i_)\} = \{z\}$. Admit that $(x_i|\ i < n)$ is not constant. Then there is an even number $l_0 \in \mathbb{Z}$ such that $x_0 = l_0(n-1)$. But this is a contradiction with $x_0 = z$. Hence $(x_i|\ i < n)$ is constant, thus $x = x_{i_0} = x_0 = z$. Therefore $\{z\}$ is closed.

Corollary 3.2. *$(\mathbb{Z}, u_n)$ is a $T_{\frac{1}{2}}$-space.*

Given a natural number $n > 1$, for each $l \in \mathbb{Z}$ we put $A_l = \{l(n-1) + i;\ i < n\}$.

Lemma 3.3. *A_l is connected in $(\mathbb{Z}, u_n)$ for each $l \in \mathbb{Z}$.*

Proof. Let $l \in \mathbb{Z}$. If l is even, then put $x_i = l(n-1)+i$ for each $i < n$, and if l is odd, then put $x_i = (l+1)(n-1)-i$ for each $i < n$. Then $(x_i|\ i<n) \in R$ and $\{x_i;\ i<n\} = A_l$. Thus, by Lemma 2.2, A_l is connected.

Theorem 3.4. *$(\mathbb{Z}, u_n)$ is a connected closure space.*

Proof. Let $(B_i|\ i<\omega)$ be the sequence given by $B_i = A_{\frac{i}{2}}$ whenever i is even and $B_i = A_{-\frac{i+1}{2}}$ whenever i is odd, i.e., $(B_i|\ i<\omega) = (A_0, A_{-1}, A_1, A_{-2}, A_2, ...)$. By Lemma 3.3, B_i is connected in $(\mathbb{Z}, u_n)$ for each $i<\omega$. For each $l \in \mathbb{Z}$ there holds $A_l \cap A_{l+1} = \{l(n-1)\} \neq \emptyset$. Thus, we have $B_0 \cap B_1 \neq \emptyset$. Let i_0 be a natural number with $i_0 > 1$. Then $B_{i_0} \cap B_{i_0-2} \neq \emptyset$ because $B_{i_0} = A_{\frac{i_0}{2}}$ and $B_{i_0-2} = A_{\frac{i_0}{2}-1}$ whenever i_0 is even, and $B_{i_0} = A_{-\frac{i_0+1}{2}}$ and $B_{i_0-2} = A_{-\frac{i_0+1}{2}+1}$ whenever i_0 is odd. Hence $\bigcup\limits_{i<i_0} B_i \cap B_{i_0} \neq \emptyset$ for each i_0, $0<i_0<\omega$. Therefore $\bigcup\limits_{i<\omega} B_i$ is connected. But $\bigcup\limits_{i<\omega} B_i = \bigcup\limits_{l\in\mathbb{Z}} A_l = \mathbb{Z}$, which proves the statement.

For any number $z \in \mathbb{Z}$ we put $L(z) = \{y \in \mathbb{Z};\ y<z\}$ and $U(z) = \{y \in \mathbb{Z};\ y>z\}$. Clearly, both $L(z)$ and $U(z)$ are closed in the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, u_n)$.

Theorem 3.5. *Let $z \in \mathbb{Z}$ be a point with $z = l(n-1)+i_0$ where $l \in \mathbb{Z}$ is an odd number and $i_0 \in \{0,-1,1\}$. Then the sets $L(z)$ and $U(z)$ are components of the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, u_n)$.*

Proof. Put $x_i = (l-1)(n-1)+i$ and $y_i = (l+1)(n-1)-i$ for each $i<n$. Then $(x_i|\ i<n) \in R_n$ and $(y_i|\ i<n) \in R_n$. Clearly, we have $z = x_{n-1} = y_{n-1}$ if $i_0 = 0$, $z = x_{n-2}$ if $i_0 = -1$, and $z = y_{n-1}$ if $i_0 = 1$. Put $D_L = \{x_i;\ i<n-1\}$ and $D_R = \{y_i;\ i<n-1\}$ if $i_0 = 0$, $D_L = \{x_i;\ i<n-2\}$ and $D_R = \{y_i;\ i<n\}$ if $i_0 = -1$, and $D_L = \{x_i;\ i<n\}$ and $D_R = \{y_i;\ i<n-2\}$ if $i_0 = 1$. Further, put $B_0 = D_L$, $C_0 = D_R$, $B_i = A_{l-i-1}$ and $C_i = A_{l+i}$ for each i, $0<i<\omega$. By Lemma 2.2, B_i and C_i are connected for each $i<i_0$. Since $B_i \cap B_{i+1} \neq \emptyset \neq C_i \cap C_{i+1}$ for each $i<\omega$, $\bigcup\limits_{i<\omega} B_i = L(z)$ and $\bigcup\limits_{i<\omega} C_i = U(z)$ are connected in $(\mathbb{Z}, u_n)$. As $L(z)$ and $U(z)$ are disjoint and closed in the subspace $\mathbb{Z}-\{z\}$ of $(\mathbb{Z}, u_n)$ and $\mathbb{Z}-\{z\} = L(z) \cup U(z)$, $L(z)$ and $U(z)$ are components of the subspace.

Remark 3.6. For $n = 2$, Theorem 3.4 states the known fact that the Khalimsky space $(\mathbb{Z}, u_n)$ is a so-called connected ordered topological space ([3]).

Lemma 3.7. *Let $z \in \mathbb{Z}$ be a point with $z = l(n-1)+i_0$ where $l \in \mathbb{Z}$ is a an even number and $i_0 \in \mathbb{Z}$ is a number fulfilling $|i_0| < n-2$. Then, in the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, u_n)$, $L(z)$ is connected whenever $i_0 > 0$, and $U(z)$ is connected whenever $i_0 < 0$.*

Proof. Let $i_0 > 0$. Then there is $(x_i|\ i<n) \in R_n$ with $x_0 = l(n-1)$ and $x_{i_0} = z$. We have $L(z) = \bigcup\{A_k;\ k \in \mathbb{Z},\ k<l\} \cup \{x_i;\ i<i_0\}$. By Lemma 2.2, A_k is connected for each $k \in \mathbb{Z}$, $k<l$, and also $\{x_i;\ i<i_0\}$ is connected. As $A_k \cap A_{k+1} \neq \emptyset$ for each $k \in \mathbb{Z}$, $k<l-1$, and also $A_{l-1} \cap \{x_i;\ i<i_0\} \neq \emptyset$, $L(z)$ is connected.

Let $i_0 < 0$. Then there is $(x_i|\ i < n) \in R_n$ with $x_0 = l(n-1)$ and $x_{i_0} = z$. We have $U(z) = \{x_i;\ i < |i_0|\} \cup \bigcup\{A_k;\ k \in \mathbb{Z},\ k \geq l\}$ and $U(z)$ is connected for the same reasons as in the case $i_0 > 0$.

Theorem 3.8. *Let $z \in \mathbb{Z}$ be a point with $z = l(n-1) + i_0$, where $l \in \mathbb{Z}$ is an even number and $i_0 \in \mathbb{Z}$ is a number fulfilling $|i_0| < n-2$, and put $B_z = \{l(n-1)+i;\ i \in \mathbb{Z},\ |i_0| < |i| < n-1,\ i_0 i \geq 0\}$. Then, in the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, u_n)$, each singleton subset of B_z is a component, and also the sets $L(z) - B_z$ and $U(z) - B_z$ are components.*

Proof. Let z fulfil the assumptions of the statement. If $i_0 \neq 0$, then there is a unique sequence $(x_i|\ i < n) \in R_n$ with $x_0 = l(n-1)$ and $x_{|i_0|} = z$. If $i_0 > 0$, then $B_z = \{x_i;\ i_0 < i < n-1\}$, so that $L(z) - B_z = L(z)$ (because $(x_i|\ i < n)$ is increasing) and $U(z) - B_z = \bigcup\{A_k;\ k \in \mathbb{Z},\ k > l\}$. If $i_0 < 0$, then $B_z = \{x_i;\ |i_0| < i < n-1\}$, so that $L(z) - B_z = \bigcup\{A_k;\ k \in \mathbb{Z},\ k < l-1\}$ and $U(z) - B_z = U(z)$ (because $(x_i|\ i < n)$ is decreasing). If $i_0 = 0$, then there are sequences $(x_i|\ i < n), (y_i|\ i < n) \in R_n$ with $x_0 = y_0 = z$ and we have $B_z = \{x_i;\ 0 < i < n-1\} \cup \{y_i;\ 0 < i < n-1\}$, thus $L(z) - B_z = \bigcup\{A_k;\ k \in \mathbb{Z},\ k < l-1\}$ and $U(z) - B_z = \bigcup\{A_k;\ k \in \mathbb{Z},\ k > l\}$. Consequently, by Proposition 3.1, each singleton subset of B_z is closed in $(\mathbb{Z}, u_n)$ and thus also in the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, u_n)$. Of course, the sets $\bigcup\{A_k;\ k \in \mathbb{Z},\ k > l\}$ and $\bigcup\{A_k;\ k \in \mathbb{Z},\ k < l-1\}$ are connected in $(\mathbb{Z}, u_n)$ (because A_k is connected for each $k \in \mathbb{Z}$ by Lemma 2.2 and $A_k \cap A_{k+1} \neq \emptyset$ for each $k \in \mathbb{Z}$, $k > l$ or $k < l-2$ respectively) and closed in $(\mathbb{Z}, u_n)$ (because l is even). Therefore, these sets are connected and closed in the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, u_n)$ because they do not contain the point z. The sets $L(z)$ and $U(z)$ are closed in the subspace $\mathbb{Z} - \{z\}$ and, by Lemma 3.7, $L(z)$ or $U(z)$ is connected in the subspace if $i_0 > 0$ or $i_0 < 0$, respectively. For any $x \in B_z$ we have $x \notin u_n(\mathbb{Z} - \{x, z\})$ and thus $\{x, z\}$ is closed in the subspace $\mathbb{Z} - \{z\}$. Hence $\{x\}$ is a component of the subspace $\mathbb{Z} - \{z\}$. Similarly, putting $C_z = \bigcup\{A_k;\ k \in \mathbb{Z},\ k > l\}$ or $C_z = \bigcup\{A_k;\ k \in \mathbb{Z},\ k < l-1\}$, for any $x \in C_z$ we have $x \notin u_n(\mathbb{Z} - (C_z \cup \{z\}))$. It follows that $\mathbb{Z} - (C_z \cup \{z\})$ is closed in the subspace $\mathbb{Z} - \{z\}$. Therefore C_z is a component of the subspace. Finally, as $\mathbb{Z} - (L(z) \cup \{z\}) = U(z)$ and $\mathbb{Z} - (U(z) \cup \{z\}) = L(z)$, both $\mathbb{Z} - (L(z) \cup \{z\})$ and $\mathbb{Z} - (U(z) \cup \{z\})$ are closed in the subspace $\mathbb{Z} - \{z\}$. Hence $L(z)$ or $U(z)$ is a component of the subspace $\mathbb{Z} - \{z\}$ if $i_0 > 0$ or $i_0 < 0$, respectively. The proof is complete.

Note that each point $z \in \mathbb{Z}$ fulfils either the assumptions of Theorem 3.5 or those of Theorem 3.8, so that the two Theorems describe the situation for each point of $\mathbb{Z}$.

4. A note on prospective applications to digital topology

Given natural numbers $m > 0$ and $n > 1$, by an *n-ary digital m-dimensional space* we understand the product of m copies of the S_n^*-space $(\mathbb{Z}, u_n)$. As for applications, the most important are the cases $m = 2$ and $m = 3$, i.e., the n-ary digital plane and the n-ary digital 3-dimensional space. Most investigations in digital topology have been done in the binary digital plane (e.g. [3], [4], [7]) and there are also some papers devoted to the study of the binary digital 3-dimensional space (e.g. [8]). In the forthcoming paper we will focus our interest at investigating the ternary digital plane $(\mathbb{Z}^2, v) = (\mathbb{Z}, u_3) \times (\mathbb{Z}, u_3)$, i.e., $v = u_{R_3 \times R_3}$. Let us note here only that in the space $(\mathbb{Z}^2, v)$ there are five classes of points that behave similarly with respect to connectedness. These classes consist of the points $x \in \mathbb{Z}^2$ for which there are precisely no, one, two, three or eight respectively *adjacent* points, i.e., points $y \in \mathbb{Z}^2 - \{x\}$ such that the set $\{x, y\}$ is connected in the space $(\mathbb{Z}^2, v)$. First of all we will try to prove an analogy of the Jordan curve theorem in $(\mathbb{Z}^2, v)$.

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