Equivalence of check digit systems over the dicyclic groups of order 8 and 12

Dedicated to Prof. Dr. Harald Scheid on occasion of his 60th birthday

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Abstrakt

Wir betrachten Prüfzeichensysteme (G, T) über einer Gruppe G mit Prüfgleichung $\prod_{i=1}^{n} T^{i}(c_i) = 1$ für Codewörter $c_1 \dots c_n \in G^n$ und anti–symmetrischer Abbildung T, d.h. einer Permutation T von G, die der Ungleichung $xT(y) \neq yT(x)$ für alle $x, y \in G$ mit $x \neq y$ genügt. Ein solches System erlaubt das Erkennen von Einzelfehlern (d.h. Fehlern in einer Komponente) und von Nachbartranspositionen ("Zahlendrehern"bei benachbarten Elementen). Zwei solche Systeme (G, T_1) und (G, T_2) heißen automorphie–äquivalent, falls $T_2 = \alpha^{-1} \circ T_1 \circ \alpha$ für ein $\alpha \in \mathrm{Aut}\, G$ gilt.

Im besonderen gehen wir bei drei speziellen Gruppen auf die anti–symmetrischen Abbildungen und ihre Automorphie–Äquivalenz–Klasen ein und berichten über die Anzahl der Elemente in den Klassen und die Fehlererkennungsraten, nämlich bei der Diedergruppe D_5 der Ordnung 10, der Quaternionen Gruppe Q_2 (der Ordnung 8) und der dizyklischen Gruppe Q_3 der Ordnung 12. Dabei ziehen wir auch Resultate von Sabine Giese und Sehpahnur Ugan heran.

Abstract

We consider check digit systems (G, T) over a group G using check equation $\prod_{i=1}^{n} T^{i}(c_i) = 1$ for codewords $c_1 \dots c_n \in G^n$ and an anti–symmetric mapping T, i.e. a permutation T of G with $xT(y) \neq yT(x)$ for all $x, y \in G$, $x \neq y$. Such a system allows to detect all single errors (i.e. errors in only one component) and neighbour transpositions (i.e. errors made by interchanging adjacent components). Two such systems (G, T_1) and (G, T_2) are called automorphism equivalent

if $T_2 = \alpha^{-1} \circ T_1 \circ \alpha$ for an automorphism α of G. In particular, we describe anti–symmetric mappings, automorphism equivalence classes and their error detection rates for three special groups, the dihedral group D_5 (of order 10), the quaternion group Q_2 (of order 8) and the dicyclic group Q_3 (of order 12); these results are part of a cooperation with Sabine Giese and Sehpahnur Ugan.

1 Introduction

To detect errors made by human operators during transmission of data, one often uses check digit systems; these usually consist of codewords $a_1 \ldots a_n$ containing, besides the information digits $a_1 \ldots a_{n-1}$, one control character a_n. In praxis, the examples used are among others the following; (see for instance [Beu95], [Sch91a], [Sch94])

- the European Article Number Code (EAN),
- the Universal Product Code (UPC),
- the International Book Number Code (ISBN) and
- the system of the serial numbers of German banknotes.

Statistical investigations of VERHOEFF [Ver69] and BECKLEY [Bec67] have shown that the most frequent errors are single errors (i.e. errors in exactly one component) and adjacent transpositions (i.e. errors made by interchanging adjacent digits). The frequencies of these and other errors in the samples mentioned are listed in Table 1.
If all codewords are of equal length insertion or deletion errors can be detected easily.

Definition
Usually a **check digit system** is defined by a group $(G, \cdot)$, the elements of which are used as the "letters" of the alphabet, and fixed permutations $\delta_1, \ldots, \delta_n$ $(n \geq 3)$ of G; then a word $a_1, \ldots a_n \in G^n$ is a codeword iff it satisfies the **check equation**

$$(1) \qquad \delta_1(a_1) \cdot \delta_2(a_2) \cdot \cdots \cdot \delta_n(a_n) = e$$

Table 1: Error types and their frequency.

error type		relative frequency in %	
		Verhoeff	Beckley
single error	$\ldots a \ldots \rightsquigarrow \ldots a' \ldots$	79.0 (60-95)	86
adjacent transposition	$\ldots a\, b \ldots \rightsquigarrow \ldots b\, a \ldots$	10.2	8
jump transposition	$\ldots acb \ldots \rightsquigarrow \ldots bca \ldots$	0.8	
twin error	$\ldots aa \ldots \rightsquigarrow \ldots bb \ldots$	0.6	6
phonetic error	$\ldots a0 \ldots \rightsquigarrow \ldots 1a \ldots$ ($a \geq 2$)	0.5	
jump twin error	$\ldots aca \ldots \rightsquigarrow \ldots bcb \ldots$	0.3	
other error		8.6	

Source: Verhoeff [Ver69](12112 pairs, 6 digits), Beckley [Bec67].

with e the neutral element of G (implicit definition).

Examples

- For the EAN, the group is $(\mathbb{Z}_{10}\ , +)$ and, for $i \in \{1, \ldots, 13\}$, δ_i is the mapping $x \mapsto w_i x$ with $w_i = 1$ for i odd and $w_i = 3$ for i even.
- For the ISBN, the group is $(\mathbb{Z}_{11}, +)$ and δ_i is $x \mapsto i \cdot x$, for $i = 1, \ldots, 10$.
- The alpha–numeric serial numbers of German banknotes are transformed into elements of the dihedral group D_5 of order 10, and $\delta_i := T_0^i$ for $i = 1, \ldots, 10$, $\delta_{11} = \mathrm{id}$ where $T_0 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 7 & 6 & 2 & 8 & 3 & 0 & 9 & 4 \end{pmatrix}$ (see for instance [Sch91a] or below).

This mapping T_0, found by VERHOEFF, is **anti–symmetric**, that means a permutation of G fulfilling the inequality

$$(2) \qquad xT(y) \neq yT(x) \quad \text{for all} \quad x, y \in G \quad \text{with} \quad x \neq y.$$

It is easy to see that a check digit system with $\delta_i := T^i$ for a permutation T of G, i. e. a system with check equation

$$(3) \qquad T(a_1)T^2(a_2) \cdots T^n(a_n) = e,$$

allows us to detect all single errors iff T is bijective and to detect all adjacent transpositions iff the permutation T is anti–symmetric, see for instance [Ver69], [EP86] or [Sch91a]. (This is a generalization of investigations by SCHAUFFLER [Schau56]. Note that some authors(see for instance [Dam98], [GM95], [Gum85], [Win90]) write the product of formula (3) in the reverse order.
In the following we consider check digit systems with anti–symmetric mappings only, so that all single errors and all adjacent transpositions can be detected. Thus, for a fixed group, the efficiency of these systems depend on the rates of detection of other non–random errors.

For reasons of comparison one can define – for a given group G – an equivalence relation on check digit systems on G with anti–symmetric mappings in such a way that all the members of each class have the same detection rates of jump transpositions, twin errors and jump twin errors. Phonetic errors are not respected by the equivalence relations considered here; they depend on the language and the actual coding of the alphabet letters by the group elements; furthermore the statistics mentioned above are not very homogeneous in the range x of the errors $x0 \rightsquigarrow 1x$.

Following the definition of the equivalence relations we shall review the situation in the special case of the dihedral group D_5 and present new results for the quaterian group Q_2 and the dicyclic group Q_3.

2 Equivalence of check digit systems

In the whole section , let G be a finite group and T_1 and T_2 be anti–symmetric mappings. Then one defines

2.1 Definition.

T_1 and T_2 are said to be **weak equivalent** if there exist elements $a, b \in G$ and an automorphism $\alpha \in \text{Aut } G$ with

$$T_2 = R_a \circ a^{-1} \circ T_1 \circ \alpha \circ L_b$$

for the rigth translation R_a with $R_a(x) = x \cdot a$ and the left translation L_b with $L_b(y) = b \cdot y$.

It may happen that weak equivalent permutations do not detect the same percentages of jump transpositions and jump twin errors; an example is given in [Gie99], 8.2.1 and in [Sch99], 6.3. Therefore we shall neglect this type of equivalence in the following and consider the other two types of equivalence relations.

2.2 Definition

(a) T_1 and T_2 (and the related check digit systems) are called **automorphism equivalent** if there exists an automorphism $\alpha \in \mathrm{Aut}\ G$ such that

$$T_2 = \alpha^{-1} \circ T_1 \circ \alpha.$$

(b) T_1 and T_2 (and the related systems) are said to be **strongly equivalent** if they are automorphism equivalent or if there exists an anti–automorphism ψ with

$$T_2 = \psi^{-1} \circ T_1^{-1} \circ \psi.$$

We mention the following properties of these equivalence relations.

2.3 Proposition

(Schulz, Giese, [Sch99], [Gie99])

(i) Automorphism–equivalence and strong equivalence of systems over G are equivalence relations.

(ii) If T_1 and T_2 are automorphism equivalent (and strongly equivalent respectively) then they detect the same percentages of jump transpositions, twin errors and jump twin errors (and, by definition, 100% of single errors and adjacent transpositions).

2.4 Situation for the group D_5

As example we recall the investigation of GIESE (cf.[Gie99],[Sch99]) on the dihedral group D_5 of order 10, the symmetry group of the regular pentagon.

Denoting by d the rotation through angle 72° around the center of the pentagon and by s a reflection on a symmetry axis, one gets

$$D_5 = \{\mathrm{id}, d, d^2, d^3, d^4, s, ds, d^2s, d^3s, d^4s\}$$

with relations $d^5 = id = s^2$ and $ds = sd^{-1}$.
One can identify the elements $d^i s^j$ of D_n with the integer $i + 5j$, so D_5 is $(\{0, \ldots, 9\}, *)$ with operation $*$ induced by the composition of symmetries of the pentagon. (For a multiplication table, see e.g. [Sch91a],[Sch99]). As mentioned in section 1, the mapping T_0 of VERHOEFF is an anti-symmetric mapping of this group. Alltogether there exist exactly 34,040 anti–symmetric mappings of D_5 ([Ver69],[Dam98],[Gie99],[Sch99]). They give rise to 1,706 equivalence classes with respect to automorphism–equivalence (GIESE by using the program package MAGMA, see [Gie99],[Sch99]) and to 911 equivalence classes concerning strong equivalence. According to the detection rates, GIESE distinguishes 7 types of automorphism equivalent classes, see Table 2.

Table 2: Types of anti–symmetric mappings of $\mathbf{D_5}$ and their detection rates in % .

	single errors and adjacent transpositions	twin errors	jump transpositions	jump twin errors	all 5 error types
Type I	100	95.56	94.22	94.22	99.90
Type II a	100	95.65	92.00	92.00	99.87
Type II b	100	91.11	94.22	94.22	99.87
Type III	100	91.11	92.00	92.00	99.84
Type IV	100	91.11	90.22	90.22	99.82
Type V	100	not of type VI and at least one rate below 90 %			99.85–99.42
Type VI	100	55.56	66.67	66.67	99.30

Source: Giese [Gie99]

Note that the relative frequency of errors used as weights for the last column of Table 2 is based on the occurrence among the non–coincidential errors (without phonetic errors) according to VERHOEFF's statistics, see Table 1.

There are two equivalence classes of Type I [Gie99], each containing 20 elements; one of the classes contains T_0, the other one the mapping

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 9 & 6 & 1 & 0 & 4 & 2 & 3 & 5 & 8 \end{pmatrix},$$

found as well by VERHOEFF [Ver69]. Type IIa, consisting of 44 classes, includes the mapping $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 5 & 7 & 6 & 1 & 2 & 4 & 9 & 3 & 8 \end{pmatrix}$ mentioned by DAMM ([Dam98]).
The examples of GUMM [Gum85], SCHULZ [Sch91b] including the mappings of BLACK [Bla72], WINTERS [Win90] and GALLIAN and MULLIN [GM95] are elements of clases of type VI (– one of the reason for the distinction of this type from Type V). The two equivalence classes of Type I fuse to one class with respect to strong equivalence; for types II to IV as well, two strong equivalence classes consist of two automorphism equivalence classes while the classes of Type VI remain unchanged.

3 The case of the dicyclic groups $\mathbf{Q}_2$ and $\mathbf{Q}_3$

Generalizing the relations which are fulfilled by the basic elements $1, i, j, k$ of the real quaternions, one defines the dicyclic groups and among these the generalized quaternion groups.

3.1 Definition and Results

(i) For $m \geq 2$, the group

$$Q_m := \langle a, b | a^{2m} = e \wedge b^2 = a^m \wedge ab = ba^{-1} \rangle$$

is called **dicyclic group**, in case $m = 2^i$ **generalized quaternion group**, for $m = 2$ just **quaternion group**.

(ii) The group Q_m has order $4m$.

(iii) By GALLIAN and MULLIN [GM95] we know that Q_m admits the following anti–symmetric mapping; (we have taken in account that they write the product of formula (3) of section 1 in reverse order).

$$\begin{array}{lll} \phi(a^i) = a^{-i}\ (0 \le i \le m-1) & \text{and} & \phi(a^i) = ba^{i-1}(m \le i \le 2m-1) \\ \phi(ba^i) = ba^{i-1}\ (0 \le i \le m-1) & \text{and} & \phi(ba^i) = a^{-i}\ (m \le i \le 2m-1). \end{array}$$

We shall now give a report on results about anti–symmetric mappings of the groups Q_2 and Q_3 which Sehpahnur Ugan [Uga99] has obtained by computer search using the programming language C^{++}. Q_3 is one of the groups beside $\mathbb{Z}_{10}$, D_5 and $\mathbb{Z}_{11}$ which has (with order 12) a size near, but not smaller than that of the set of decimal digits. The results on Q_2 are (like the verification of the case D_5) a by–product of testing the programs.

3.2 Situation of the quaternion group

a) **Multiplication and coding**

By coding the elements of $Q_2 = \langle a, b | a^4 = e \wedge b^2 = a^2 \wedge ab = ba^{-1} \rangle$ by $a^i b^j \longmapsto i + 4j$ $(i = 0, \ldots, 3;\ j = 0, 1)$ one gets the following multiplication table.

Table 3: Multiplication table of the quaternion group.

*	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	2	1	0	3
5	5	4	7	6	3	2	1	0
6	6	5	4	7	0	3	2	1
7	7	6	5	4	1	0	3	2

b) **Automorphism of $\mathbf{Q_2}$**

The elements of Q_2 except $e(\hat{=}0)$ and $a^2(\hat{=}2)$ all have order 4. So the orbit of $a(\hat{=}1)$ under Aut Q_2 has at most length 6, and the stabilizer of a in Aut Q_2 can map $b(\hat{=}4)$ to $a^i b$ only $(i = 0, \ldots, 3)$. Hence $|\text{Aut } Q_2| \leq 24$. Indeed, all these automorphisms exist, hence $|\text{Aut } Q_2| = 24$ and $|\text{Antiaut } Q_2| = 48$.

c) **Anti–symmetric mappings of $\mathbf{Q_2}$**

There exist exactly 1,152 anti–symmetric mappings of Q_2 (S. Ugan [Uga99]), which constitute 48 equivalence classes of size 24 each with respect to automorphism equivalence; concerning strong equivalence, there are 12 classes à 24 and 18 classes à 48 elements. For Q_2, S. Ugan distinguishes 5 types of anti-symmetric mappings according to their error detection rate, see Table 4.

Table 4: Types of check digit systems of $\mathbf{Q_2}$ and their detection rates in % .

	Type 1	Type 2	Type 3	Type 4	Type 5	Σ
single errors	100%	100%	100%	100%	100%	
adjacent transpositions	100%	100%	100%	100%	100%	
twin errors	85.71	85.71	78.57	78.57	78.57	
jump transpositions	92.86	$<$ 89.29	92.86	89.29	$<$ 89.29	
jump twin errors	92.86	$<$ 89.29	92.86	89.29	$<$ 89.29	
All 5 error types (weighted according to Verhoeff's statistics	99.82	99.73–99.65	99.77	99.73	99.69–99.60	
number of automorphism equivalence classes	12	4	4	8	20	48
number of strong equivalence classes	7	3	2	4	14	30
number of check digit systems	288	96	96	192	480	1,152

Source: Ugan [Uga99].

d) **Representatives for Type 1**

We list here representatives of the 7 classes of Type 1 with respect to strong equivalence

(in the form $[x_0 \ldots x_7]$ for $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{pmatrix}$

[0 2 4 6 3 7 5 1]
[1 0 4 7 6 2 3 5]
[1 3 4 6 2 0 7 5]
[1 4 0 5 3 6 2 7]
[1 4 0 7 5 3 6 2]
[1 4 2 5 6 0 7 3]
[1 4 6 3 0 5 7 2]

The mapping found by GALLIAN & MULLIN is, in our terminology, [0 3 7 6 5 1 2 4] and automorphism equivalent to [0 2 4 6 3 7 5 1].

3.3 The dicyclic group of order 12

a) **Multiplication and coding**

If one puts $m = 3$ in 3.1(i) one gets the dicyclic group

$$Q_3 = \langle a, b | a^6 = e \wedge b^2 = a^3 \wedge ab = ba^{-1} \rangle$$

of order 12. The elements of this group can be coded by the numbers 0 to 11 by

$$a^i b^j \longmapsto i + 6j \quad (i = 0, \ldots, 5; \quad j = 0, 1).$$

This yields the multiplicaton table

$i * j$	$0 \leq j \leq 5$	$6 \leq j \leq 11$
$0 \leq i \leq 5$	$(i + j)$MOD 6	$(i + j)$MOD $6 + 6$
$6 \leq i \leq 11$	$(i - j)$MOD $6 + 6$	$(i - j + 3)$MOD 6

,

or, in full, Table 5.

Table 5: **Multiplication table of Q_3.**

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	0	7	8	9	10	11	6
2	2	3	4	5	0	1	8	9	10	11	6	7
3	3	4	5	0	1	2	9	10	11	6	7	8
4	4	5	0	1	2	3	10	11	6	7	8	9
5	5	0	1	2	3	4	11	6	7	8	9	10
6	6	11	10	9	8	7	3	2	1	0	5	4
7	7	6	11	10	9	8	4	3	2	1	0	5
8	8	7	6	11	10	9	5	4	3	2	1	0
9	9	8	7	6	11	10	0	5	4	3	2	1
10	10	9	8	7	6	11	1	0	5	4	3	2
11	11	10	9	8	7	6	2	1	0	5	4	3

b) **Automorphisms of Q_3**

Any automorphism of Q_3 must fix the sets $\{a, a^5\}$ of elements of order 6 and $\{a^ib|i = 0, \ldots, 5\}$ of elements of order 4. So there can exist at most 12 automorphisms; but

$$\alpha_{rs} : a^ib^j \longmapsto a^{ri}(a^sb)^j \text{ for } i = 0, \ldots, 5 \text{ and } j = 0, 1$$

with $r \in \{1, 5\}$ and $s \in \{0, \ldots, 5\}$ define automorphisms; thus $|\text{Aut } Q_3|$ $= 12$ and $|\text{Antiaut } Q_3| = 24$, since [0 1 2 3 4 5 6 11 10 9 8 7] is an antiautomorphism.

c) **Anti–symmetric mappings of Q_3**

According to computer search by S. UGAN ([Uga99]) there are exactly 1,403,136 anti–symmetric mappings of Q_3; (this means that only approximately 0.3% of the 12! permutations of Q_3 are anti–symmetric). The best detection rates are as follows.

– 41,472 of these mappings have a detection rate of twin errors of 96.97% (128 out of 132); they form 3,456 equivalence classes with respect to automorphism equivalence and 1,728 strong equivalence classes.

– 144 mappings (in 12 automorphism eqivalence classes and 6 strong equivalence classes) allow to detect 96.21% (1524 of 1584) of jump transpositions.

In the following, there are only considered mappings detecting more than 90% of each of the error types twin errors, jump transpositions and jump twin errors; these are 329,760 in 27,480 automorphism equivalence classes of size 12 or in 13,740 strong equivalence classes of size 24. According to the weighted detection rates, S. UGAN distinguishes between 4 types of these anti–symmetric mappings, see Table 6.

Table 6: Types of check digit systems of $\mathbf{Q_3}$ with detection rates of over 90% for each considered error type.

	Type I	Type II	Type III	Type IV
single errors	100%	100%	100%	100%
adjacent transpositions	100%	100%	100%	100%
twin errors	96.97%	96.97–93,94	96.97–93.94	96.97– 90.91
jump transpositions	94.70%	95.71–93.94	95.96–92.42	96.21 –90.15
jump twin errors	94.70%	95.45–93.18	95.96–90.66	96.21 –90.15
all 5 error types (weighted)	99.92%	99.91	99.90	99.89–99.82
number of automorphism equivalence classes	8	204	804	26,464
number of strong equivalence classes	4	102	402	13,232
number of check digit systems	96	2,448	9,648	317,568

Source: S. Ugan [Uga99]

d) **Representatives for Type I**

Here are the representatives of the 4 classes of Type I with respect to strong equivalence

(in the form $[x_0 \dots x_{11}]$ for $\begin{pmatrix} 0 & 1 & \dots & 11 \\ x_0 & x_1 & & x_{11} \end{pmatrix} \in \mathrm{Sym}(Q_3)$).

$$\begin{bmatrix} 1 & 6 & 5 & 7 & 0 & 11 & 9 & 4 & 2 & 10 & 8 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 5 & 6 & 8 & 10 & 4 & 11 & 9 & 1 & 3 & 0 & 7 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 2 & 4 & 11 & 7 & 3 & 8 & 0 & 10 & 5 & 9 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 3 & 10 & 4 & 11 & 2 & 1 & 5 & 9 & 7 & 0 & 8 \end{bmatrix}$$

e) **The Mapping of Gallian and Mullin**

For $m = 3$, the anti–symmetric mapping defined in 3.1(iii) is

$$[\ 0\ \ 5\ \ 4\ \ 10\ \ 9\ \ 8\ \ 7\ \ 1\ \ 2\ \ 3\ \ 11\ \ 6\ ]\ ;$$

it has a detection rate of 81,82% for twin errors (so it is not contained in Table 6), of 92,42% for jump transpositions and jump twin errors respectively; the weighted rate for all 5 errors under consideration is 99,79% (cf. [Uga99]).

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