HIGHER SEPARATION AXIOMS, PARACOMPACTNESS AND DIMENSION FOR SEMIUNIFORM CONVERGENCE SPACES

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Dedicated to Professor Lamar Bentley on his 60th birthday

Abstract. Higher separation axioms, paracompactness and dimension are defined for semiuniform convergence spaces. Under the assumption that subspaces are formed in the construct **SUConv** of semiuniform convergence spaces one obtains the following results: 1^0 Subspaces of normal (symmetric) topological spaces are normal, 2^0 subspaces of paracompact topological spaces are paracompact, and 3^0 for some dimension functions, the dimension of a subspace is less than or equal to the dimension of the original space, where these dimension functions coincide with the (Lebesgue) covering dimension for paracompact topological spaces. Additionally, Urysohn's Lemma, Tietze's extension theorem and some other extension theorems are studied not only in the realm of topological spaces. Last but not least the interrelations between nearness spaces and semiuniform convergence spaces become apparent.

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0. Introduction

It is well–known that the construct **Top** of topological spaces has several deficiencies, e.g. there do not exist natural function spaces in general (i.e. **Top** is not cartesian closed) and quotients are not hereditary (cf. [7; Thm. 2]). Furthermore, the formation of subspaces in **Top** is not satisfactory as the following *example* shows: Though a point or the closed point interval $[0, 1]$ are distinguishable (they are non–homeomorphic), the topological spaces obtained from the real line $\mathbb{R}$ by removing a point or $[0, 1]$ are not distinguishable (they are homeomorphic). Additionally, uniform concepts such as uniform continuity, completeness or uniform convergence cannot be explained in the framework of topological spaces. In Convenient Topology (cf. [16]) all these deficiencies are remidied by introducing semiuniform convergence spaces (cf. [15]). In particular, one obtains non–isomorphic spaces in the above example provided that subspaces are formed in the construct **SUConv** of semiuniform convergence spaces instead of **Top**. This better behaviour of subspaces in **SUConv** leads to much nicer results even for classical topological concepts such as paracompactness, normality or covering dimension as will be explained in the following:

According to Katětov [10] filter spaces can be described in the framework of merotopic spaces. On the other hand, filter spaces have also been described in the realm of semiuniform convergence spaces (cf. [15]). Thus, they form the link between semiuniform convergence spaces and merotopic spaces. From this fact profit the definitions of complete regularity, normality, full normality (resp. paracompactness) and dimension for semiuniform convergence spaces which are introduced in this article (complete regularity of filter spaces has been studied earlier in [2] and [3]). It turns out that all higher separation axioms studied here are hereditary and that for the small and the large filter–dimension introduced here, the dimension of a subspace is less than or equal to the dimension of the original space, where these dimension functions coincide with the (Lebesgue) covering dimension for paracompact topological spaces.

Additionally, Urysohn's Lemma and Tietze's extension theorem are stated for normal filter spaces, and an extension theorem for Cauchy continuous (resp. uniformly continuous) maps into spheres is related to the small filter dimension (resp. small uniform dimension) of normal filter spaces (resp. uniform spaces).

Nearness spaces introduced by Herrlich [6] form an important special case of merotopic spaces and several results on them are needed for the above mentioned theorems on semiuniform convergence spaces.

The terminilogy of this article corresponds to [1] and [13].

1. Preliminaries

For the convenience of the reader some basic definitions are repeated.
A *semiuniform convergence space* is a pair $(X, \mathcal{J}_X)$, where X is a set and $\mathcal{J}_X$ a set of filters on $X \times X$ such that 1^0 $\dot{x} \times \dot{x} \in \mathcal{J}_X$ for each $x \in X$ (where $\dot{x} = \{A \subset X : x \in A\}$), 2^0 $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{F} \subset \mathcal{G}$, and 3^0 $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\} \in \mathcal{J}_X$ (where $F^{-1} = \{(y,x) : (x,y) \in F\}$). A map $f : (X, \mathcal{J}_X) \longrightarrow (Y, \mathcal{J}_Y)$ between semiuniform convergence spaces is called *uniformly continuous* provided that $(f \times f)(\mathcal{F}) \in \mathcal{J}_Y$ for each $\mathcal{F} \in \mathcal{J}_X$. The construct of semiuniform convergence spaces is denoted by **SUConv**.

A uniform space $(X, \mathcal{V})$ in the sense of Weil [21] may be considered to be a semiuniform convergence space $(X, \mathcal{J}_X)$ with $\mathcal{J}_X = [\mathcal{V}]$ where $[\mathcal{V}] = \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset \mathcal{V}\}$ and $F(X \times X)$ denotes the set of all filters on $X \times X$. In this case $(X, \mathcal{J}_X)$ is also called a *principal uniform limit space.* A symmetric topological space (= R_0–space) $(X, \mathcal{X})$ may be considered to be a semiuniform convergence space $(X, \mathcal{J}_X)$, where $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there is some $x \in X$ with $\mathcal{F} \supset \mathcal{U}_\mathcal{X}(x) \times \mathcal{U}_\mathcal{X}(x)\}$ and $\mathcal{U}_\mathcal{X}(x)$ denotes the neighborhood filter of $x \in X$ w.r.t. $\mathcal{X}$. In this case $(X, \mathcal{J}_X)$ is called a *topological semiuniform convergence space.*

Every semiuniform convergence space $(X, \mathcal{J}_X)$ has an *underlying filter space* $(X, \gamma_{\mathcal{J}_X})$, where $\gamma_{\mathcal{J}_X} = \{\mathcal{F} \in F(X) : \mathcal{F} \times \mathcal{F} \in \mathcal{J}_X\}$ and $F(X)$ denotes the set of all filters on X (remember that a filter space is a pair (X, γ), where X is a set and $\gamma \subset F(X)$ contains all filters $\dot{x}$ with $x \in X$ and for each $\mathcal{F} \in \gamma$, all filters $\mathcal{G}$ on X with $\mathcal{F} \subset \mathcal{G}$; a map $f : (X, \gamma) \longrightarrow (X', \gamma')$ between filter spaces is called *Cauchy continuous* provided that $f(\mathcal{F}) \in \gamma'$ for each $\mathcal{F} \in \gamma$). Furthermore, a filterspace (X, γ) may be considered to be a semiuniform convergence space $(X, \mathcal{J}_\gamma)$ where $\mathcal{J}_\gamma = \{\mathcal{F} \in F(X \times X)$: there is some $\mathcal{G} \in \gamma$ with $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$. In particular, a semiuniform convergence $(X, \mathcal{J}_X)$ is called **Fil**–*determined* provided that $\mathcal{J}_X = \mathcal{J}_{\gamma_{\mathcal{J}_X}}$.

A *symmetric Kent convergence space* is a pair (X,q) where X is a set and $q \subset F(X) \times X$ such that 1^0 $(\dot{x}, x) \in q$ for each $x \in X$, 2^0 $(\mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{G} \supset \mathcal{F}$, 3^0 $(\mathcal{F} \cap \dot{x}, x) \in q$ whenever $(\mathcal{F}, x) \in q$, and 4^0 $(\mathcal{F}, x) \in q$ and $y \in \bigcap\{F : F \in \mathcal{F}\}$ imply $(\mathcal{F}, y) \in q$ (symmetry condition). A map $f : (X, q) \longrightarrow (X', q')$ between (symmetric) Kent convergence spaces is called *continuous* provided that $(f(\mathcal{F}), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$. The construct of symmetric Kent convergence spaces is denoted by $\mathbf{KConv}_\mathrm{S}$. Every semiuniform convergence space$(X, \mathcal{J}_X)$ has an *underlying symmetric Kent convergence space* $(X, q_{\gamma_{\mathcal{J}_X}})$ defined by $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$ iff $\mathcal{F} \cap \dot{x} \in \gamma_{\mathcal{J}_X}$. Furthermore, a symmetric Kent convergence space (X, q) may be considered to be a semiuniform convergence space $(X, \mathcal{J}_{\gamma_q})$ where $\gamma_q = \{\mathcal{F} \in F(X)$: there is some $x \in X$ with $(\mathcal{F}, x) \in q\}$.

For every symmetric Kent convergence space (X, q) a closure operator cl_q is defined by $cl_q A = \{x \in X$: there is some $\mathcal{F} \in F(X)$ with $(\mathcal{F}, x) \in q$ and $A \in \mathcal{F}\}$ for each subset A of X; occasionally, one writes $\overline{A}$ instead of $cl_q A$. In particular, $\mathcal{X}_q = \{O \subset X : cl_q(X \backslash O) = X \backslash O\}$ is a topology on X. For each topological space $(X, \mathcal{X})$, define $q_\mathcal{X} \subset F(X) \times X$ by $(\mathcal{F}, x) \in q_\mathcal{X}$ iff $\mathcal{F} \supset \mathcal{U}_\mathcal{X}(x)$. A symmetric Kent convergence space (X, q) is called *topological* provided that $q = q_{\mathcal{X}_q}$.

A subset A of a semiuniform convergence space $(X, \mathcal{J}_X)$ is called *closed* (resp. *dense*) provided that $A = cl_{q_{\gamma_{\mathcal{J}_X}}} A$ (resp. $cl_{q_{\gamma_{\mathcal{J}_X}}} A = X$). A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *regular* provided that for each $\mathcal{F} \in \mathcal{J}_X$ the subfilter*) $\overline{\mathcal{F}}$ generated by the filter base $\{\overline{F} : F \in \mathcal{F}\}$ belongs to $\mathcal{J}_X$, where $\overline{F}$ denotes the closure of F in the underlying Kent convergence space of the product space $(X, \mathcal{J}_X) \times (X, \mathcal{J}_X)$ (note that **SUConv** is a topological construct!). The underlying symmetric Kent convergence space $(X, q_{\gamma_{\mathcal{J}_X}})$ of a regular semiuniform convergence space $(X, \mathcal{J}_X)$ is regular, i.e. for each $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$, $(\overline{\mathcal{F}}, x) \in q_{\gamma_{\mathcal{J}_X}}$ where $\overline{\mathcal{F}}$ is generated by $\{\overline{F} : F \in \mathcal{F}\}$ and $\overline{F} = cl_{q_{\gamma_{\mathcal{J}_X}}} F$; furthermore, a symmetric topological space $(X, \mathcal{X})$ is regular in the usual sense iff it is regular as a semiuniform convergence space. Using [5; 3.7.13. and 4.12.6.] one obtains the following

Theorem. *Let $(X, \mathcal{J}_X)$ be a principal uniform limit space (resp. a topological semiuniform convergence space). Then each uniformly continuous map $f : (A, \mathcal{J}_A) \longrightarrow (Y, \mathcal{J}_Y)$ from a dense subspace $(A, \mathcal{J}_A)$ of $(X, \mathcal{J}_X)$ to a complete, regular and separated semiuniform convergence space $(Y, \mathcal{J}_Y)$ has a unique uniformly continuous extension $\overline{f} : (X, \mathcal{J}_X) \longrightarrow (Y, \mathcal{J}_Y)$.*

(Note: A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *complete* provided that for each $\mathcal{F} \in \gamma_{\mathcal{J}_X}$ there is some $x \in X$ such that $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$; and it is called *separated* (or a T_2*–space*) provided that for each $\mathcal{F} \in F(X)$ there is at most one $x \in X$ such that $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$.)

By the way, a semiuniform convergence space $(X, \mathcal{J}_X)$ is called a T_1*–space* provided that for each pair $(x, y) \in X \times X$, $(\dot{x}, y) \in q_{\gamma_{\mathcal{J}_X}}$ implies $x = y$. Obviously, every regular T_1–space (= T_3*–space*) is a T_2–space (cf. [5; 3.7.12.]).
Concerning nearness spaces and generalizations the reader is referred to [13], where in the following we say merotopic space instead of seminearness space; in particular, **Mer**, **Near**, **Fil**, **Top**, **Top**$_\mathrm{S}$ and **Unif** stand for the constructs of merotopic spaces, nearness spaces, filter spaces, topological spaces, R_0–spaces and uniform spaces respectively.

2. Complete regularity

2.1 Definitions. 1) A filter space (X, γ) is called *completely regular* provided that for each $\mathcal{F} \in \gamma$ the subfilter $\mathcal{G} = \{G \subset X : F$ is completely within G for some $F \in \mathcal{F}\}$ belongs to γ; here F is completely within G provided that there is a Cauchy

*)A subset $\mathcal{G}$ of a filter $\mathcal{F}$ on a set X is called a *subfilter* provided that it is a filter.

continuous map $f : (X, \gamma) \longrightarrow ([0,1], \gamma_t)$ such that $f[F] \subset \{0\}$ and $f[X \backslash G] \subset \{1\}$, where γ_t denotes the set of all convergent filters on the closed unit interval $[0,1]$ endowed with the usual topology.

2) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *completely regular* provided that it is regular and its underlying filter space $(X, \gamma_{\mathcal{J}_X})$ is completely regular.

2.2 Remarks. 1) Each filter space (X, γ) has a corresponding (filter–)merotopic space (X, μ_γ), where $\mu_\gamma = \{\mathcal{A} \subset \mathcal{P}(X)$: for each $\mathcal{F} \in \gamma$ there is some $A \in \mathcal{A}$ with $A \in \mathcal{F}\}$ (cf. [17; 2.4.]), i.e. there is an alternative description of filter spaces in the realm of merotopic spaces.

2) *A filter space* (X, γ) *is completely regular iff* (X, μ_γ) *is a completely regular nearness space* (cf. [2] and [3]).

3) According to [3] (resp. [2]) *a filter space is completely regular iff it is a subspace* (*in* **Fil**) *of some completely regular topological space* (*regarded as a filter space*) [note that there is no difference in forming subspaces of symmetric topological spaces in **Mer**, **Fil** or **Near** respectively]. Consequently, each completely regular filter space is subtopological; furthermore, it is a Cauchy space [note that every competely regular topological space is weakly Hausdorff (cf. [5; 3.7.12]) and that is is a Cauchy space (cf. [14; 2.13]); furthermore **Chy** is bireflective in **Fil** (cf. [14; 3.1])].

2.3 Proposition. *The underlying Kent convergence space of a completely regular semiuniform convergence space is a completely regular topological space* (*in the usual sense*).

Proof. Let $(X, \mathcal{J}_X)$ be a completely regular semiuniform convergence space. Then $(X, \gamma_{\mathcal{J}_X})$ is a completely regular filter space, i.e. there is a completely regular topological space $(Y, \mathcal{Y})$ such that $(X, \gamma_{\mathcal{J}_X})$ is a subspace (in **Fil**) of $(Y, \gamma_{q_\mathcal{Y}})$, where $\gamma_{q_\mathcal{Y}}$ consists of all convergent filters in $(Y, \mathcal{Y})$. Since initial structures in **Fil** induce initial structures in $\mathbf{KConv}_\mathrm{S}$, $(X, q_{\gamma_{\mathcal{J}_X}})$ is a subspace (in $\mathbf{KConv}_\mathrm{S}$) of $(Y, q_\mathcal{Y})$ [$(\mathcal{F}, y) \in q_\mathcal{Y}$ iff $\mathcal{F}$ converges to y in $(Y, \mathcal{Y})$]. Since $(Y, q_\mathcal{Y})$ is a completely regular topological space and $\mathbf{Top}_\mathrm{S}$ is closed under formation of subspaces in $\mathbf{KConv}_\mathrm{S}$, $(X, q_{\gamma_{\mathcal{J}_X}})$ is a completely regular topological space.

2.4 Corollary. *A symmetric Kent convergence space* (X, q) *is completely regular* (*as a semiuniform convergence space*) *iff it is a competely regular topological space.*

Proof. "$\Longrightarrow$". If $(X, \mathcal{J}_{\gamma_q})$ is completely regular, then, by 2.3., $(X, q_{\gamma_{\mathcal{J}_{\gamma_q}}}) = (X, q)$ is a completely regular topological space.

"$\Longleftarrow$". If (X, q) is a competely regular topological space, then $(X, \mathcal{J}_{\gamma_q})$ is a regular semiuniform convergence space (cf. [15; 5.6.a)]). Furthermore, $(X, \gamma_{\mathcal{J}_{\gamma_q}}) = (X, \gamma_q)$ is completely regular by 2.2.3).

2.5 Proposition. *Every uniform space is completely regular.*

Proof. By [15; 5.9.], every uniform space is regular. Furthermore, if $(X, \mathcal{V})$ is a uniform space and $(X^*, \mathcal{V}^*)$ a complete uniform space containing $(X, \mathcal{V})$ as a dense subspace (cf. e.g. [19; 3.4., Satz 1]), then the underlying filter space (= Cauchy space) $(X, \gamma_{\mathcal{V}})$ of $(X, \mathcal{V})$ is a subspace in **Fil** of the underlying filter space (= Cauchy space) $(X^*, \gamma_{\mathcal{V}^*})$ of $(X^*, \mathcal{V}^*)$. Since $(X^*, \mathcal{V}^*)$ is complete, $(X^*, \gamma_{\mathcal{V}^*})$ is the corresponding filter space of the underlying completely regular topological space of $(X^*, \mathcal{V}^*)$. Thus, $(X, \gamma_{\mathcal{V}})$ is a completely regular filter space (cf. 2.2.3)). Therefore, everything is proved.

2.6 Proposition. *Every completely regular filter space is regular* (*as a semiuniform convergence space*).

Proof. Let (X, γ) be a completely regular filter space. Since (X, γ) is a subspace (in **Fil**) of a completely regular topological space, which (considered to be a semiuniform convergence space) is a regular semiuniform convergence space, (X, γ) (regarded as a semiuniform convergence space) is also regular because the regular semiuniform convergence spaces form a bireflective subconstruct of **SUConv** and subspaces in **Fil** are formed as in **SUConv**.

2.7 Remark. The two preceding propositions demonstrate that for important examples of semiuniform convergence spaces whose underlying filter spaces are completely regular the regularity is automatically fulfilled. But this is not always the case as the following *example* shows: Let$X$ be the set $I\!R$ of real numbers and $\mathcal{X}$ the usual topology on $I\!R$. Define $A \subset \Delta_X$ by $(x, x) \in A$ iff $0 < x < 1$ and let $\mathcal{U}_{\mathcal{X}}(x)$ be the neighborhood filter of $x \in X$ with respect to $\mathcal{X}$.
Put $\mathcal{J}_X = \{\mathcal{F} \in F(I\!R^2) : \mathcal{F} \supset (\{A\})\} \cup \{\mathcal{F} \in F(I\!R^2) : \mathcal{F} \supset \mathcal{U}_{\mathcal{X}}(x) \times \mathcal{U}_{\mathcal{X}}(x)$ for some $x \in I\!R\}$. Then $(X, \mathcal{J}_X)$ is a semiuniform convergence space such that the underlying filter space $(X, \gamma_{\mathcal{J}_X})$ is completely regular (obviously, $\gamma_{\mathcal{J}_X} = \{\mathcal{F} \in F(I\!R) : \mathcal{F}$ converges in $(X, \mathcal{X})\}$), but $(X, \mathcal{J}_X)$ is not regular since $(\{A\})$ belongs to $\mathcal{J}_X$ whereas the subfilter $\overline{(\{A\})} = (\{\overline{A}\})$ does not belong to $\mathcal{J}_X$.

2.8 Proposition. *The construct* **CReg** *of completely regular semiuniform convergence spaces (and uniformly continuous maps) is a bireflective subconstruct of* **SU-Conv** *and thus a topological construct.*

Proof. Let $(f_i : (X, \mathcal{J}_X) \longrightarrow (X_i, \mathcal{J}_{X_i}))_{i \in I}$ be an initial source in **SUConv** such that all $(X_i, \mathcal{J}_{X_i})$ are completely regular. Then $(X, \mathcal{J}_X)$ is completely regular:

1) By [15; 5.3], $(X, \mathcal{J}_X)$ is regular.
2) By [15; 3.10], $\gamma_{\mathcal{J}_X}$ is the initial **Fil**–structure w.r.t. (f_i), and by [3; 3.2], $(X, \gamma_{\mathcal{J}_X})$ is a completely regular filter space.

2.9 Remark. By 2.8., every subspace (in **SUConv**) of a completely regular topological space (regarded as a semiuniform convergence space) is competely regular. It follows from 2.5. that the inverse is not true, namely the uniform space $I\!R_u$ of real numbers is completely regular but it is not a subspace (in **SUConv**) of a competely regular topological space [otherwise it would be **Fil**–determined and thus it might be considered to be the topological space $I\!R_t$ of real numbers, which is impossible since the uniform structure of $I\!R_u$ is not indiscrete (cf. [15; 5.10.])].

3. Normality

3.1 Definitions. 1) A merotopic space (= seminearness space) (X, μ) is called

a) *regular* provided that the following is satisfied:

(R) For each $\mathcal{U} \in \mu$, there is some (refinement) $\mathcal{V} \in \mu$ such that for each $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ with $\{X \backslash V, U\} \in \mu$,
b) *normal* provided that (X, μ) and (X, μ_c) are regular where $\mu_c = \{\mathcal{U} \in \mu$: there is some finite $\mathcal{V} \in \mu$ with $\mathcal{V} \prec \mathcal{U}\}$

2) Let (X, γ) be a filter space and (X, μ_γ) its corresponding (filter–)merotopic space. Then (X, γ) is called *merotopically normal* (shortly: *m–normal*) provided that (X, μ_γ) is normal.

3) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *normal* provided that it is regular and the underlying filter space $(X, \gamma_{\mathcal{J}_X})$ is m–normal.

3.2 Remark. Every regular merotopic space is a nearness space (cf. e.g. [13; 6.2.7. ①]).

3.3 Proposition. *Every normal semiuniform convergence space is completely regular.*

Proof. Let $(X, \mathcal{J}_X)$ be a normal semiuniform convergence space. Then $(X, \mathcal{J}_X)$ is regular. Furthermore, $(X, \mu_{\gamma_{\mathcal{J}_X}})$ is a nearness space which is normal and thus completely regular (cf. [2]). Consequently, by 2.2.2), $(X, \gamma_{\mathcal{J}_X})$ is completely regular.

3.4 Proposition. *Let* $\mathbf{X}$ *be a filter space. Then the following are equivalent:*

(1) $\mathbf{X}$ *is normal* (*as a semiuniform convergence space*)
(2) $\mathbf{X}$ *is* m*–normal.*

Proof. (1) $\Longrightarrow$ (2). This follows immediately from the definitions.
(2) $\Longrightarrow$ (1). Since $\mathbf{X}$ coincides with its underlying filter space whenever $\mathbf{X}$ is considered to be a semiuniform convergence space, it suffices to prove that $\mathbf{X}$ is regular. Since $\mathbf{X}$ is normal as a merotopic space, it is also a completely regular nearness space. Consequently, by 2.2.2), the filter space $\mathbf{X}$ is completely regular. Thus, by 2.6., $\mathbf{X}$ is regular as a semiuniform convergence space.

3.5 Proposition. *A symmetric Kent convergence space is normal* (*as a semiuniform convergence space*) *iff it is a normal topological space in the usual sense.*

Proof. 1) Let $(X, \mathcal{X})$ be a symmetric topological space, $(X, \gamma_{q_\mathcal{X}})$ its corresponding filter space (i.e. $\gamma_{q_\mathcal{X}}$ is the set of all convergent filters in $(\mathcal{X}, \mathcal{X})$) and $(X, \mu_\mathcal{X})$ its corresponding nearness space, i.e. $\mu_\mathcal{X} = \{\mathcal{U} \subset \mathcal{P}(X)$: there is some open cover $\mathcal{O}$ of $(X, \mathcal{X})$ with $\mathcal{O} \prec \mathcal{U}\}$. Then $\mu_{\gamma_{q_\mathcal{X}}} = \mu_\mathcal{X}$:

a) Let $\mathcal{U} \in \mu_{\gamma_{q_\mathcal{X}}}$. Thus, each convergent filter in $(X, \mathcal{X})$ contains some $U \in \mathcal{U}$. Hence, $\{U^0 : U \in \mathcal{U}\} \prec \mathcal{U}$ and is an open cover of X; namely if $x \in X$, the neighborhood filter $\mathcal{U}_\mathcal{X}(x)$ of x in $(X, \mathcal{X})$ converges to x and thus there is some $U \in \mathcal{U}$ such that $U \in \mathcal{U}_\mathcal{X}(x)$, i.e. $x \in U^0 \subset U$. Therefore, $\mathcal{U} \in \mu_\mathcal{X}$.

b) Let $\mathcal{U} \in \mu_\mathcal{X}$, i.e. there is an open cover $\mathcal{O}$ of $(X, \mathcal{X})$ with $\mathcal{O} \prec \mathcal{U}$. If $\mathcal{F}$ is a filter on X such that there is some $x \in X$ with $\mathcal{F} \supset \mathcal{U}_\mathcal{X}(x)$, then there are some $O \in \mathcal{O}$ and some $U \in \mathcal{U}$ such that $x \in O \subset U$. Thus, $U \in \mathcal{U}_\mathcal{X}(x) \subset \mathcal{F}$. Consequently, $\mathcal{U} \in \mu_{\gamma_{q_\mathcal{X}}}$.

2) a) Let (X, q) be a symmetric Kent convergence space such that $(X, \mathcal{J}_{\gamma_q})$ is normal. Then $(X, \mathcal{J}_{\gamma_q})$ is completely regular and by 2.3., (X, q) is a (completely regular) topological space, i.e. $q = q_{\mathcal{X}_q}$. By assumption, $(X, \gamma_{\mathcal{J}_{\gamma_q}}) = (X, \gamma_q)$ is m–normal, i.e. $(X, \mu_{\mathcal{X}_q})$ is normal (cf. 1)). By [13; 7.22.], $(X, \mathcal{X}_q)$ (resp. (X, q)) is a normal

topological space in the usual sense.

b) Let (X,q) be a normal symmetric topological space in the usual sense, i.e. $\mathcal{X}_q$ is a normal symmetric topology on X. In order to prove that $(X,\mathcal{J}_{\gamma_q})$ is normal it suffices to show that (X,μ_{γ_q}) is normal since (X,γ_q) is a filter space and 3.4. is valid. By 1), $(X,\mu_{\gamma_q})=(X,\mu_{\mathcal{X}_q})$. Since $(X,\mathcal{X}_q)$ is a normal symmetric topological space, it is regular and by [8; 4.3.5], $(X,\mu_{\mathcal{X}_q})$ is regular. Furthermore, $(X,(\mu_{\mathcal{X}_q})_c)$ is regular by [13; 7.2.2.]. Thus, $(X,\mu_{\mathcal{X}_q})$ is normal and the proof is finished.

3.6 Remarks. 1) *The underlying Kent convergence space of a normal semiuniform convergence space need not be normal* as the following *example* shows (whereas it is always a completely regular topological space by 3.3. and 2.3.): Let $(X,\mathcal{X})$ be a completely regular Hausdroff space, which is not normal (e.g. the Niemytzki plane), and (X,μ_F) the corresponding fine uniform space described by means of uniform covers, i.e. μ_F is the set of all covers of X which are refined by some normally open cover of $(X,\mathcal{X})$. Further, consider $(X,(\mu_F)_c)$ and let $\mathcal{W}_{(\mu_F)_c}$ denote the Weil uniformity corresponding to $(\mu_F)_c$. Then $(X,\mathcal{W}_{(\mu_F)_c})$ is a totally bounded uniform space which is separated since its underlying topological space is the Hausdorff space $(X,\mathcal{X})$. Consequently, $(X,[\mathcal{W}_{(\mu_F)_c}])$ is a normal semiuniform convergence space (cf. 4.11. and 4.2.), whose underlying Kent convergence space $(X,q_{\mathcal{X}})$ is not normal.

2) *A uniform space need not be normal*, namely in the following an *example* of a complete separated uniform space is given which is not normal: Let $I\!R_u$ be the normal uniform space of real numbers and $I\!R_u^{I\!R}$ the uniform product space. Let $\mathcal{V}$ be the uniformity of $I\!R_u^{I\!R}$ and put $\mathcal{J}_{I\!R^{I\!R}}=[\mathcal{V}]$, i.e. $(I\!R^{I\!R},\mathcal{J}_{I\!R^{I\!R}})$ is the principal uniform limit space corresponding to $I\!R_u^{I\!R}$. Then the set $\gamma_{\mathcal{J}_{I\!R^{I\!R}}}$ of its Cauchy filters is given by

$$(*)\qquad \gamma_{\mathcal{J}_{I\!R^{I\!R}}}=\{\mathcal{F}\in F(I\!R^{I\!R}):\mathcal{F}\ \text{ converges in } I\!R_t^{I\!R}\},$$

where $I\!R_t$ denotes the normal topological space of real numbers and the topological product space $I\!R_t^{I\!R}$ is the underlying Kent convergence space of $(I\!R^{I\!R},\mathcal{J}_{I\!R^{I\!R}})$. Further,

$$(**)\qquad \mu_{\gamma_{\mathcal{J}_{I\!R^{I\!R}}}}=\mu_{\mathcal{X}_{\mathcal{V}}},$$

where $\mathcal{X}_{\mathcal{V}}$ denotes the topology induced by $\mathcal{V}$, i.e. the topology of $I\!R_t^{I\!R}$ (cf. part 1) of the proof of 3.5.). Since $I\!R_t^{I\!R}$ is not normal in the usual topological sense (cf. [22; 21 C.5.], $(I\!R^{I\!R},\mu_{\mathcal{X}_{\mathcal{V}}})$ cannot be normal (cf. [13; 7.2.2.]). By $(**)$, this implies that the complete separated uniform space $I\!R_u^{I\!R}$ is not normal as a semiuniform convergence space.

3.7 Proposition. *Every subspace* (*in* **SUConv**) *of a normal semiuniform convergence space is normal.*

Proof. Let $(X, \mathcal{J}_X)$ be a normal semiuniform convergence space and $U \subset X$. If $\mathcal{J}_u$ denotes the initial **SUConv**–structure on U w.r.t. the inclusion map $i : U \longrightarrow X$, then $(U, \mathcal{J}_u)$ is regular (cf. [15; 5.3.]). By [15; 3.10], $(U, \gamma_{\mathcal{J}_u})$ is a subspace (in **Fil**) of $(X, \gamma_{\mathcal{J}_X})$. Since subspaces in **Fil–Mer** ($\cong$ **Fil**) are formed as in **Mer**, $(U_{\mu_{\gamma_{\mathcal{J}_u}}})$ is a subspace of $(X, \mu_{\gamma_{\mathcal{J}_X}})$. Then $(U, \mu_{\gamma_{\mathcal{J}_u}})$ is a normal nearness space because $(X, \mu_{\gamma_{\mathcal{J}_X}})$ is a normal nearness space (cf. [13; 6.2.7. ⑤ and 7.2.10. ①]).

3.8 Remarks. 1) As is well–known subspaces in **Top** (resp. $\mathbf{Top}_{\mathrm{S}}$) of normal symmetric topological spaces need not be normal in general. But if subspaces are formed in **SUConv** one obtains from 3.7. the following result: *Subspaces of normal symmetric topological spaces are normal.*

2) The class of all normal semiuniform convergence spaces does not coincide with the class of all subspaces (in **SUConv**) of normal symmetric topological spaces as the following *example* shows: Modify the example under 2.7. by defining $A \subset \Delta_X$ as follows:

$$(x, x) \in A \text{ iff } 0 \leq x \leq 1.$$

Then $(X, \mathcal{J}_X)$ is a normal semiuniform convergence space which is not a subspace of a normal topological semiuniform convergence space since it is not **Fil**–determined.

3) Since products of symmetric topological spaces are formed in $\mathbf{Top}_{\mathrm{S}}$ as in **SUConv** and in $\mathbf{Top}_{\mathrm{S}}$ normality is not finitely productive, it follows from 3.5. that in **SUConv** *normality is not finitely productive.*

3.9 Definition. Let (X, γ) be a filter space. Then $\mathcal{A} \subset \mathcal{P}(X)$ is called *near* provided that there is some $\mathcal{F} \in \gamma$ such that for each $A \in \mathcal{A}$, $X \setminus A \notin \mathcal{F}$.

3.10 Theorem (Urysohn's Lemma). *Let* (X, γ) *be a normal filter space. Whenever* $\{A, B\} \subset \mathcal{P}(X)$ *is not near, there is a Cauchy continuous map* $f : (X, \gamma) \longrightarrow ([0, 1], \gamma_t)$ *such that* $f[A] \subset \{0\}$ *and* $f[B] \subset \{1\}$ *where* γ_t *denotes the set of all convergent filters w.r.t. the usual topology on the closed unit interval* $[0, 1]$.

Proof. Cf. [13; 7.2.6] (resp. [18]) and note:

1. $\{A, B\} \subset \mathcal{P}(X)$ is not near in (X, γ) iff $\{A, B\}$ is not near in (X, μ_γ),

i.e. $\{X\backslash A, X\backslash B\} \in \mu_\gamma$.
2. $f : (X,\gamma) \longrightarrow ([0,1],\gamma_t)$ is Cauchy continuous iff $f : (X,\mu_\gamma) \longrightarrow ([0,1],\mu_{\gamma_t})$ is uniformly continuous (μ_{γ_t} coincides with the fine uniform structure on $[0,1]$).

3.11 Theorem (Tietze, Urysohn). *Let (X,γ) be a normal filter space, and (A,γ_A) a subspace (in* **Fil***) of (X,γ). Then every Cauchy continuous map $f : (A,\gamma_A) \longrightarrow ([0,1],\gamma_t)$ has a Cauchy continuous extension $F : (X,\gamma) \longrightarrow ([0,1],\gamma_t)$.*

Proof. Cf. [13; 7.2.13] (resp. [18]) and note that subspaces of filter spaces are formed in **Fil** ($\cong$ **Fil–Mer**) as in **Mer**.

3.12 Remark. For normal R_0–spaces, the classical version of Urysohn's Lemma follows from 3.10 (note: $A = \overline{A}$, $B = \overline{B}$ and $A \cap B = \emptyset$ imply $\{A,B\}$ is not near) and that one of Tietze's extension theorem is an immediate consequence of 3.11 (note: every closed subspace of an R_0–space is a subspace in **Fil**).

4. Full normality and paracompactness

4.1 Definitions. 1) A merotopic space (X,μ) is called *uniform* provided that each $\mathcal{A} \in \mu$ is star–refined by some $\mathcal{B} \in \mu$.

2) A filter space (X,γ) is called *merotopically uniform* (shortly: *m–uniform*) provided that (X,μ_γ) is uniform.

3) A semiuniform convergence space $(X,\mathcal{J}_X)$ is called *fully normal* provided that it is regular and its underlying filter space $(X,\gamma_{\mathcal{J}_X})$ is m–uniform.

4) A semiuniform convergence space $(X,\mathcal{J}_X)$ is called *paracompact* provided that it is T_1 and fully normal.

4.2 Proposition. *Every fully normal semiuniform convergence space is normal.*

Proof. Let $(X,\mathcal{J}_X) \in |\,\mathbf{SUConv}\,|$ be fully normal. It suffices to prove that $(X,\gamma_{\mathcal{J}_X})$ is m–normal. But this is obvious, since every uniform space (as a merotopic space) is normal (cf. e.g. [13; 6.2.7. 2 a) and 3.1.3.8. 3]).

4.3 Corollary. *Every paracompact semiuniform convergence space is T_4, i.e. normal and T_1.*

4.4 Proposition. *A filter space is fully normal* (*as a semiuniform convergence space*) *iff it is m–uniform.*

Proof. "$\Longrightarrow$". If $(X,\gamma)\in|\mathbf{Fil}|$ such that $(X,\mathcal{J}_\gamma)$ is fully normal, then $(X,\gamma_{\mathcal{J}_\gamma})=(X,\gamma)$ is m–uniform.

"$\Longleftarrow$". If $(X,\gamma)\in|\mathbf{Fil}|$ is m–uniform, then (X,μ_γ) is uniform and thus it is normal. By 3.4., $(X,\mathcal{J}_\gamma)$ is normal and consequently it is regular. Furthermore, $(X,\gamma_{\mathcal{J}_\gamma})=(X,\gamma)$ is m–uniform by assumption.

4.5 Proposition. *A symmetric Kent convergence space is fully normal* (*as a semiuniform convergence space*) *iff it is a fully normal topological space in the usual sense.*

Proof. "$\Longrightarrow$". Let (X,q) be a symmetric Kent convergence space such that $(X,\mathcal{J}_{\gamma_q})$ is fully normal. Then $(X,\mathcal{J}_{\gamma_q})$ is completely regular and by 2.3., (X,q) is a (completely regular) topological space, i.e. $q=q_{\mathcal{X}_q}$. By assumption, $(X,\gamma_{\mathcal{J}_{\gamma_q}})=(X,\gamma_q)$ is m–uniform, i.e. $(X,\mu_{\mathcal{X}_q})$ is uniform (cf. part 1) of the proof of 3.5.). By [13; 3.1.2.7.], $(X,\mathcal{X}_q)$ (resp. (X,q))) is fully normal.

"$\Longleftarrow$". Let (X,q) be a fully normal R_0–space in the usual sense, i.e. $\mathcal{X}_q$ is a fully normal symmetric topology on X. In order to prove that $(X,\mathcal{J}_{\gamma_q})$ is fully normal it suffices to show that (X,μ_{γ_q}) is uniform since (X,γ_q) is a filter space and 4.4. is valid. We know already that $(X,\mu_{\gamma_q})=(X,\mu_{\mathcal{X}_q})$. Since $(X,\mathcal{X}_q)$ is a fully normal R_0–space, $(X,\mu_{\mathcal{X}_q})$ is uniform.

4.6 Corollary. *A symmetric topological space is paracompact* (*as a semiuniform convergence space*) *iff it is paracompact in the usual sense.*

4.7 Definition. A filter space (X,γ) is called *topological* provided that it is weakly subtopological (i.e. (X,q_γ) is topological) and complete.

4.8 Remark. The construct **TopFil** of topological filter spaces (and Cauchy continuous maps) is (concretely) isomorphic to $\mathbf{Top}_{\mathrm{S}}$.

4.9 Proposition. *A symmetric topological space is fully normal iff it is topological and m–uniform* (*as a filter space*).

Proof. "$\Longrightarrow$". Let $(X,\mathcal{X})$ be a fully normal R_0–space and $(X,\gamma_{q_\mathcal{X}})$ the corresponding filter space. Obviously, $(X,\gamma_{q_\mathcal{X}})$ is complete; and since $q_{\gamma_{q_\mathcal{X}}}=q_\mathcal{X}$, $(X,\gamma_{q_\mathcal{X}})$

is subtopological, i.e. $(X, \gamma_{q_\mathcal{X}})$ is topological. Furthermore, $(X, \mu_{\gamma_{q_\mathcal{X}}}) = (X, \mu_\mathcal{X})$ is uniform, since $(X, \mathcal{X})$ is fully normal. Thus, $(X, \gamma_{q_\mathcal{X}})$ is m–uniform.

"$\Longleftarrow$". Let $(X, \mathcal{X})$ be an R_0–space such that $(X, \gamma_{q_\mathcal{X}})$ is (topological and) m–uniform. By 4.4., $(X, \mathcal{J}_{\gamma_{q_\mathcal{X}}})$ is fully normal, and by 4.5., $(X, q_\mathcal{X})$ (resp. $(X, \mathcal{X})$) is fully normal.

4.10 Corollary. *A topological T_1–space is paracompact iff it is topological and m–uniform (as a filter space).*

4.11 Proposition. *Every proximity space (= totally bounded uniform space) is fully normal.*

Proof. Let $(X, \mathcal{J}_X)$ be a proximity space, i.e. $\mathcal{J}_X = [\mathcal{V}]$ where $\mathcal{V}$ is a totally bounded uniformity on X. Since $(X, \mathcal{J}_X)$ is uniform, it is regular (cf. [15; 5.9.]). It remains to show that $(X, \gamma_{\mathcal{J}_X})$ is m–uniform. Let $(X, \mu_\mathcal{V})$ be the merotopic space corresponding to $\mathcal{V}$, i.e. $\mu_\mathcal{V} = \{\mathcal{A} \subset \mathcal{P}(X)$: there is some $V \in \mathcal{V}$ with $\mathcal{A}_V \prec \mathcal{A}\}$, where $\mathcal{A}_V = \{V(x) : x \in X\}$. If $\gamma_{\mu_\mathcal{V}}$ denotes the set of all Cauchy filters in $(X, \mu_\mathcal{V})$ (resp. $(X, \mathcal{V})$), then

(1) $\gamma_{\mu_\mathcal{V}} = \gamma_{\mathcal{J}_X}$.

Since $(X, \mu_\mathcal{V})$ is a contigual (= totally bounded = precompact) merotopic space, it is filtermerotopic (cf. [10; 2.10]) and thus $\mu_{\gamma_{\mu_\mathcal{V}}} = \mu_\mathcal{V}$ (cf. part c) in the proof of [17; 2.4]). Consequently, by (1), one obtains:

(2) $\mu_{\gamma_{\mathcal{J}_X}} = \mu_\mathcal{V}$.

Since $(X, \mu_\mathcal{V})$ is uniform, it follows from (2) that $(X, \gamma_{\mathcal{J}_X})$ is m–uniform.

4.12 Corollary. *Every separated proximity space is paracompact.*

4.13 Proposition. *A complete uniform space is fully normal (as a semiuniform convergence space) iff its underlying topological space is fully normal.*

Proof. "$\Longleftarrow$": Let $(X, \mathcal{V})$ be a complete uniform space such that its underlying topological space $(X, \mathcal{X}_\mathcal{V})$ is fully normal. Put $\mathcal{J}_X = [\mathcal{V}]$. Since $(X, \mathcal{V})$ is complete, $\mu_{\gamma_{\mathcal{J}_X}} = \mu_{\gamma_{q_{\mathcal{X}_\mathcal{V}}}}$ and since $(X, \mathcal{X}_\mathcal{V})$ is fully normal, $\mu_{\gamma_{\mathcal{J}_X}} = \mu_{\gamma_{q_{\mathcal{X}_\mathcal{V}}}} = \mu_{\mathcal{X}_\mathcal{V}}$ is a uniform structure in the sense of Tukey. Furthermore, $(X, \mathcal{J}_X)$ is regular. Thus, $(X, \mathcal{J}_X)$ is fully normal.

"$\Longrightarrow$". Let $(X, \mathcal{V})$ be a complete uniform space such that $(X, [\mathcal{V}])$ is fully normal.

Put $\mathcal{J}_X = [\mathcal{V}]$. By assumption, $(X, \mu_{\gamma_{\mathcal{J}_X}})$ is uniform. Since $\mu_{\gamma_{\mathcal{J}_X}} = \mu_{\mathcal{X}_\mathcal{V}}$, $(X, \mathcal{X}_\mathcal{V})$ is fully normal.

4.14 Remarks. 1) By 4.13., the uniform space $I\!R_u$ of real numbers is paracompact as a semiuniform convergence space. Furthermore, the topological space $I\!R_t$ of real numbers is paracompact as a semiuniform convergence space (cf. 4.6.).

2) A complete uniform space need not have an underlying topological space which is fully normal as the example under 3.6.2) shows.

3) *The underlying Kent convergence space of a fully normal semiuniform convergence space need not be fully normal*; namely, there is a separated proximitiy space whose underlying topological space is not paracompact: Consider a completely regular Hausdorff space which is not paracompact (c.g. the ordinal space Ω_0 of all ordinals less than the first uncountable ordinal ω_1). Continuing as under 3.6. 1) leads to the desired example.

4.15 Proposition. *Every subspace* (*in* **SUConv**) *of a fully normal* (*resp. paracompact*) *semiuniform convergence space is fully normal* (*resp. paracompact*).

Proof. Let $(X, \mathcal{J}_X)$ be a fully normal semiuniform convergence space, $U \subset X$ and $\mathcal{J}_U$ the initial **SUConv**–structure w.r.t. the inclusion map $i : U \longrightarrow X$:

a) $(U, \mathcal{J}_U)$ is regular since $(X, \mathcal{J}_X)$ is regular (cf. [15; 5.3.]).
b) $(U, \gamma_{\mathcal{J}_U})$ is a subspace (in **Fil**) of $(X, \gamma_{\mathcal{J}_X})$. Since subspaces in **Fil** ($\cong$ **Fil**–**Mer**) are formed as in **Mer**, $(U, \mu_{\gamma_{\mathcal{J}_U}})$ is a subspace (in **Mer**) of $(X, \mu_{\gamma_{\mathcal{J}_X}})$. Since the construct **Unif** of uniform spaces (and uniformly continuous maps) is bireflective in **Mer**, $(U, \mu_{\gamma_{\mathcal{J}_U}})$ is uniform, i.e. $(U, \gamma_{\mathcal{J}_U})$ is m–uniform.
It follows from a) and b) that $(U, \mathcal{J}_U)$ is fully normal.
Since obviously subspaces of T_1–spaces are T_1–spaces, the above proposition is also valid for paracompactness.

4.16 Remarks. 1) As is well–known subspaces in **Top** (resp. $\mathbf{Top}_\mathrm{S}$) of paracompact topological spaces need not be paracompact. But if subspaces are formed in **SUConv** one obtains from 4.15. the following result: *Subspaces of paracompact topological spaces are paracompact.*

2) A paracompact semiuniform convergence space need not be a subspace (in **SUConv**) of a paracompact topological space; namely the example under 3.8.2) is a counterexample (indeed, $(X, \mathcal{J}_X)$ is paracompact!). Together with 4.15. one obtains

that the class of semiuniform convergence spaces which are subspaces of paracompact topological spaces is a proper subclass of the class of paracompact semiuniform convergence spaces.

3) Since *paracompactness* is not finitely productive in $\mathbf{Top}_{\mathrm{S}}$, it *is* also *not finitely productive in* **SUConv** (cf. the corresponding result for normality under 3.8.3)).

4) Full normality is related to the other higher separation axioms by means of the following implication scheme:

$$
\begin{tikzcd}
\boxed{\text{proximity space}} \arrow[r, Rightarrow] \arrow[drr, Rightarrow] & \text{fully normal} \arrow[r, Rightarrow] & \text{normal} \arrow[r, Rightarrow] & \text{competely regular} \arrow[r, Rightarrow] & \text{regular} \\
& & \boxed{\text{uniform space}} \arrow[ur, Rightarrow] & &
\end{tikzcd}
$$

Adding the T_1–axiom one obtains from the above implication scheme the following one:

$$
\begin{tikzcd}
\boxed{\text{separated proximity space}} \arrow[r, Rightarrow] \arrow[drr, Rightarrow] & \text{paracompact} \arrow[r, Rightarrow] & T_4 \arrow[r, Rightarrow] & T_{3\frac{1}{2}} \arrow[r, Rightarrow] & T_3 \arrow[r, Rightarrow] & T_2 \arrow[r, Rightarrow] & T_1 \\
& & \boxed{\text{separated uniform space}} \arrow[ur, Rightarrow] & & & &
\end{tikzcd}
$$

5. Dimension functions

5.1 Definitions. A) Let (X, μ) be a merotopic space.
1) The *large dimension* Dim (X, μ) of (X, μ) is said to be $\leq n$ provided that every uniform cover $\mathcal{U}$ of X has a refinement $\mathcal{V} \in \mu$ of order $\leq n + 1$ (i.e. each $x \in X$ is contained in at most $n + 1$ elements of $\mathcal{V}$). The precise number Dim (X, μ) is the smallest such n, or -1 for the special case that X is empty; and we write Dim $(X, \mu) = \infty$ if there is no such n.

2) The *small dimension* dim (X, μ) of (X, μ) is defined to be the large dimension of (X, μ_c). Especially, dim $(X, \mu) \leq n$ iff every finite uniform cover $\mathcal{U}$ of X has a (finite) refinement $\mathcal{V} \in \mu$ of order $\leq n + 1$.

B) Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space.
1) a) The *large filter–dimension* $\mathrm{Dim}_f(X, \mathcal{J}_X)$ of $(X, \mathcal{J}_X)$ is defined to be

Dim $(X, \mu_{\gamma_{\mathcal{J}_X}})$.
b) The *small filter–dimension* $\dim_f(X, \mathcal{J}_X)$ of $(X, \mathcal{J}_X)$ is defined to be dim $(X, \mu_{\gamma_{\mathcal{J}_X}})$.

2) a) The *large uniform dimension* $\mathrm{Dim}_u(X, \mathcal{J}_X)$ of $(X, \mathcal{J}_X)$ is defined to be the large dimension of $(X, \mu_{\mathcal{V}})$ where $(X, [\mathcal{V}])$ denotes the underlying uniform space of $(X, \mathcal{J}_X)$.
b) The *small uniform dimension* $\dim_u(X, \mathcal{J}_X)$ of $(X, \mathcal{J}_X)$ is defined to be the small dimension of $(X, \mu_{\mathcal{V}})$ (cf. a)).

5.2 Remarks. 1) a) For symmetric topological spaces $(X, \mathcal{X})$, one writes dim $(X, \mathcal{X})$ instead of dim $(X, \mu_{\mathcal{X}})$ (resp. Dim $(X, \mathcal{X})$ instead of Dim $(X, \mu_{\mathcal{X}})$), where $(X, \mu_{\mathcal{X}})$ denotes the merotopic space correponding to $(X, \mathcal{X})$. Then dim $(X, \mathcal{X})$ coincides with the (Lebesgue) covering dimension of $(X, \mathcal{X})$ (cf. [12; definition I.4])
b) For paracompact topological spaces $(X, \mathcal{X})$, dim $(X, \mathcal{X})$ = Dim $(X, \mathcal{X})$ (cf. [11; 9–14]).

2) a) Let $(X, \mathcal{V})$ be a uniform space. Then dim $(X, \mu_{\mathcal{V}}) = \dim_u(X, [\mathcal{V}])$ coincides with Isbell's uniform dimension $\delta d\,(X, \mu_{\mathcal{V}})$ and Dim $(X, \mu_{\mathcal{V}}) = \mathrm{Dim}_u(X, [\mathcal{V}])$ is identical with Isbell's large dimension $\Delta d\,(X, \mu_{\mathcal{V}})$ (cf. [9]).
b) $\dim_u(X, [\mathcal{V}]) = \mathrm{Dim}_u(X, [\mathcal{V}])$ provided that $\mathrm{Dim}_u(X, [\mathcal{V}]) < \infty$ (cf. [9]).

5.3 Proposition. 1) *If* $(X, \mathcal{X})$ *is a symmetric topological space, then* $\dim_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ = dim $(X, \mathcal{X})$ *and* $\mathrm{Dim}_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ = Dim $(X, \mathcal{X})$.

2) *If* $(X, \mathcal{X})$ *is a paracompact topological space, then additionally,* $\dim_u(X, \mathcal{J}_{\gamma_{q\mathcal{X}}}) = \dim_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}}) = \mathrm{Dim}_u(X, \mathcal{J}_{\gamma_{q\mathcal{X}}}) = \mathrm{Dim}_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$.

Proof. 1) $(X, \gamma_{q\mathcal{X}})$ is the underlying filter space of $(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$, where $\gamma_{q\mathcal{X}}$ consists of all convergent filters on $(X, \mathcal{X})$. Hence, $(X, \mu_{\gamma_{q\mathcal{X}}}) = (X, \mu_{\mathcal{X}})$ which implies the desired equalities.

2) Since $(X, \mathcal{X})$ is paracompact, there is a finest uniformity $\mathcal{V}$ on X which induces $\mathcal{X}$ and it is easily checked that $(X, [\mathcal{V}])$ is the underlying uniform space of $(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$. Then $\mu_{\mathcal{V}} = \mu_{\mathcal{X}}$. This implies (together with 1)) that $\dim_u(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ = dim $(X, \mu_{\mathcal{V}})$ = dim $(X, \mu_{\mathcal{X}}) = \dim_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ = dim $(X, \mathcal{X})$ and $\mathrm{Dim}_u(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ = Dim $(X, \mu_{\mathcal{V}})$ = Dim $(X, \mu_{\mathcal{X}}) = \mathrm{Dim}_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ = Dim $(X, \mathcal{X})$. Consequently, the desired result follows from 5.2.1) b).

5.4 Remark. For normal symmetric topological spaces $(X, \mathcal{X})$, $\mathrm{Dim}_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ may differ from $\dim_f(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$, e.g. if $(X, \mathcal{X}) = \Omega_0$, then dim $\Omega_0 = 0$ and Dim $\Omega_0 = \infty$ (cf. [8; 5.4.8]).

5.5 Proposition. *Let $(X, \mathcal{V})$ be a proximity space (= totally bounded uniform space). Then*

$\dim_u(X, [\mathcal{V}]) = \dim_f(X, [\mathcal{V}]) = \dim\,(X, \mu_{\mathcal{V}}) = \mathrm{Dim}\,(X, [\mathcal{V}]) =$
$= \mathrm{Dim}_f(X, [\mathcal{V}]) = \mathrm{Dim}\,(X, \mu_{\mathcal{V}})$.

Proof. Since $\mu_{\gamma_{[\mathcal{V}]}} = \mu_{\mathcal{V}}$ (cf. (2) in the proof of 4.11), $\dim_u(X, [\mathcal{V}]) = \dim\,(X, \mu_{\mathcal{V}}) = \dim\,(X, \mu_{\gamma_{[\mathcal{V}]}}) = \dim_f(X, [\mathcal{V}])$ and $\mathrm{Dim}_u(X, [\mathcal{V}]) = \mathrm{Dim}\,(X, \mu_{\mathcal{V}}) = \mathrm{Dim}_f(X, [\mathcal{V}])$. Furthermore, $\mathrm{Dim}\,(X, \mu_{\mathcal{V}}) = \dim\,(X, \mu_{\mathcal{V}})$ because $(\mu_{\mathcal{V}})_c = \mu_{\mathcal{V}}$.

5.6 Remark. For a given proximity space the common value of all dimension functions considered above is known to be Smirnov's δ–dimension of it (cf. [20]).

5.7 Proposition. *If $(X, \mathcal{V})$ is a complete uniform space, then* $\dim_f(X, [\mathcal{V}]) = \dim\,(X, \mathcal{X}_{\mathcal{V}})$ *and* $\mathrm{Dim}_f(X, [\mathcal{V}]) = \mathrm{Dim}\,(X, \mathcal{X}_{\mathcal{V}})$, *where $(X, \mathcal{X}_{\mathcal{V}})$ denotes the underlying topological space of $(X, \mathcal{V})$.*

Proof. Since by assumption $\mu_{\gamma_{[\mathcal{V}]}} = \mu_{\mathcal{X}_{\mathcal{V}}}$, the desired result is obvious.

5.8 Corollary. $\mathrm{Dim}_u I\!R^n_t = \mathrm{Dim}_f I\!R^n_t = \mathrm{Dim}\, I\!R^n_t = \mathrm{Dim}_f I\!R^n_u = \mathrm{Dim}_u I\!R^n_u = \dim\, I\!R^n_t = \dim_f I\!R^n_t = \dim_u I\!R^n_t = \dim_f I\!R^n_u = \dim_u I\!R^n_u = n$.

Proof. Since $I\!R^n_t$ is a paracompact topological space, it follows from 5.3. that

$\dim_u I\!R^n_t = \dim_f I\!R^n_t = \dim\, I\!R^n_t = \mathrm{Dim}_u I\!R^n_t = \mathrm{Dim}_f I\!R^n_t = \mathrm{Dim}\, I\!R^n_t$,

where $\dim\, I\!R^{\,n}_{\,t} = n$ (cf. e.g. [4; 7.3.19]). Since $I\!R^{\,n}_{\,u}$ is a complete uniform space, it follows from 5.7. that $\dim_f I\!R^n_u = \dim\, I\!R^n_t$ and $\mathrm{Dim}_f I\!R^n_u = \mathrm{Dim}\, I\!R^n_t$.
Furthermore, $\dim_u I\!R^n_u = \mathrm{Dim}_u I\!R^n_u = n$ (cf. [9; V.12]). Thus, the corollary is proved.

5.9 Proposition. *If $(X, \mathcal{J}_X)$ is a semiuniform convergence space, then the following are valid:*

a) $\dim_f(X, \mathcal{J}_X) \leq \mathrm{Dim}_f(X, \mathcal{J}_X)$,
b) $\dim_u(X, \mathcal{J}_X) \leq \mathrm{Dim}_u(X, \mathcal{J}_X)$.

Proof. a) and b) follow immediately from

$$\dim\,(X, \mu) \leq \mathrm{Dim}\,(X, \mu)$$

for each merotopic space (X, μ) (cf. [8; 5.4.7]).

5.10 Proposition. 1) *Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space and $(A, \mathcal{J}_A)$ a subspace (in* **SUConv**) *of $(X, \mathcal{J}_X)$. Then the following are satisfied:*

a) $\mathrm{Dim}_f(A, \mathcal{J}_A) \leq \mathrm{Dim}_f(X, \mathcal{J}_X)$,

b) $\dim_f(A, \mathcal{J}_A) \leq \dim_f(X, \mathcal{J}_X)$.

2) *Let $(X, \mathcal{J}_X)$ be a principal uniform limit space (= uniform space) and $(A, \mathcal{J}_A)$ a subspace (in* **SUConv**) *of $(X, \mathcal{J}_X)$. Then the following are satisfied:*

a) $\mathrm{Dim}_u(A, \mathcal{J}_A) \leq \mathrm{Dim}_u(X, \mathcal{J}_X)$,

b) $\dim_u(A, \mathcal{J}_A) \leq \dim_u(X, \mathcal{J}_X)$.

Proof. 1) and 2) follow immediately from the fact that for each merotopic space (X, μ) and each subspace (A, μ_A) (in **Mer**) of (X, μ) the following are valid:

a) $\mathrm{Dim}\ (A, \mu_A) \leq \mathrm{Dim}\ (X, \mu)$,

b) $\dim\ (A, \mu_A) \leq \dim\ (X, \mu)$

(cf. [8; 5.4.10]).

5.11 Corollary. *Let* **X** *be a symmetric topological space and* **A** *a closed subspace in* **Top** *(resp.* $\mathbf{Top}_{\mathrm{S}}$*). Then*

$$\dim \mathbf{A} \leq \dim \mathbf{X} \text{ and } \mathrm{Dim}\ \mathbf{A} \leq \mathrm{Dim}\ \mathbf{X}.$$

Proof. Since closed subspaces in $\mathbf{Top}_{\mathrm{S}}$ are formed as in **SUConv** the desired result follows from 5.10.1) and 5.3.1).

5.12 Proposition. 1) *Let $(X, \mathcal{X})$ be a regular topological space and $(A, \mathcal{J}_A)$ a dense subspace (in* **SUConv**) *of $(X, \mathcal{J}_{\gamma_{q_{\mathcal{X}}}})$. Then*

$$\mathrm{Dim}_f(X, \mathcal{J}_{\gamma_{q_{\mathcal{X}}}}) = \mathrm{Dim}_f(A, \mathcal{J}_A)\,.$$

2) *Let $(X, \mathcal{J}_X)$ be a principal uniform limit space (= uniform space) and $(A, \mathcal{J}_A)$ a dense subspace (in* **SUConv**)*. Then*

$$\mathrm{Dim}_u(X, \mathcal{J}_X) = \mathrm{Dim}_u(A, \mathcal{J}_A).$$

Proof. 1) and 2) follow from the fact that for each regular merotopic space (X, μ) and each dense subspace (A, μ_A) of (X, μ), $\mathrm{Dim}\ (A, \mu_A) = \mathrm{Dim}\ (X, \mu)$ (cf. [8; 5.4.11.]).

5.13 Theorem. 1) *Let (X, γ) be a normal filter space. Then $\dim_f(X, \mathcal{J}_\gamma) \leq n$ iff every Cauchy continuous map of any subspace (A, γ_A) (in* **Fil**) *of (X, γ) into the n–sphere S^n has a Cauchy continuous extension over (X, γ), where S^n is endowed with* *its usual topological* **Fil**–*structure consisting of all convergent filters w.r.t. the usual topology of* S^n.

2) *Let* $(X, \mathcal{J}_X)$ *be a principal uniform limit space (= uniform space). Then* $\dim_u(X, \mathcal{J}_X) \leq n$ *iff every uniformly continuous map of any subspace* $(A, \mathcal{J}_A)$ *(in* **SUConv**) *of* $(X, \mathcal{J}_X)$ *into the* n–*sphere* S^n *has a uniformly continuous extension over* $(X, \mathcal{J}_X)$, *where* S^n *is endowed with its usual uniform structure.*

Proof. For every normal filter space (X, γ), (X, μ_γ) is a normal nearness space, and for every uniform space $(X, [\mathcal{V}])$, $(X, \mu_\mathcal{V})$ is a normal nearness space. Thus, the above theorem follows from [13; 7.2.17.] note that the corresponding merotopic structure of the usual topological **Fil**–structure of S^n is the usual uniform structure).

5.14 Corollary. *Let* $(X, \mathcal{X})$ *be a normal* R_0–*space. Then* $\dim\,(X, \mathcal{X}) \leq n$ *iff every continuous map of any closed subspace* $(A, \mathcal{X}_A)$ *(in* **Top**) *of* $(X, \mathcal{X})$ *into the* n–*sphere* S^n *has a continuous extension over* $(X, \mathcal{X})$.

5.15 Remark. For normal **Fil**–determined semiuniform convergence spaces $(X, \mathcal{J}_X)$ of finite small filter dimension $\dim_f(X, \mathcal{J}_X) = \dim\,(X, \mu_{\gamma_{\mathcal{J}_X}})$, $\dim_f(X, \mathcal{J}_X)$ can be characterized cohomologically (cf. [13; 7.3.3.] and note that a pair of filter spaces is a pair $(\mathbf{X}, \mathbf{A})$, where $\mathbf{X} = (X, \gamma)$ is a filter space and $\mathbf{A} = (A, \gamma_A)$ is a subspace (in **Fil**) of $\mathbf{X}$, and $\check{H}_f^n(\mathbf{X}, \mathbf{A})$ is defined to be $\check{H}_f^n((X, \mu_\gamma), (A, \mu_{\gamma_A}))$. Similarly, for uniform spaces (= principal uniform limit spaces) $(X, [\mathcal{V}])$ of finite large uniform dimension $\mathrm{Dim}_u(X, [\mathcal{V}]) = \mathrm{Dim}\,(X, \mu_\mathcal{V})$, $\mathrm{Dim}_u(X, [\mathcal{V}])$ can be characterized cohomologically (cf. [13; 7.3.3.] and note that for pairs of uniform spaces $((X, \mathcal{V}), (A, \mathcal{V}_A))$, $\check{H}_f^n((X, \mathcal{V}), (X, \mathcal{V}_A))$ is defined to be $\check{H}_f^n((X, \mu_\mathcal{V}), (A, \mu_{\mathcal{V}_A})))$.

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