Divisible designs admitting, as an automorphism group, an orthogonal group or a unitary group

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Abstract

We construct some divisible designs starting from a projective space. These divisible designs admit an orthogonal group or a unitary group as an automorphism group.

Keywords: divisible designs, R-permutation groups, automorphism groups, orthogonal group, unitary group, constant weight codes. **Classification**: 05B30, 51E20, 51E15, 20B25, 20G40, 94B60.

1 Introduction

A method to construct divisible designs by t-R-homogeneous (t-R-transitive) imprimitive permutation groups (G, X, R) was introduced by A.G. Spera in [Sp1]. The divisible designs which are constructed by such a method admit a large automorphism group. Imprimitive groups which are 2-R-homogeneous (2-R-transitive) were obtained from translation planes that have a collineation group operating 2-transitively on the line at infinity.(See for example [SchSp1],[SchSp2], [CeSp], [Ce].) In this work we introduce a method to construct 2-R-transitive imprimitive permutation groups from the desarguesian 3-space, giving rise to the construction of divisible designs. In particular, we consider the affine space $\mathrm{AG}(3,q)$, with ideal plane E and its translation group T, and we choose in E a set of lines Ω admitting a subgroup G of the collineation group of E which acts 2-transitively. We embed G in $\mathrm{PGL}(4,q)$, and call the embedded group $\widetilde{G}$; then $T \cdot \widetilde{G}$ acts 2-R-transitively on $\Pi(\Omega) = \{\pi \text{ a plane of } \mathrm{AG}(3,q) \mid \pi \cap E \in \Omega\}$, if R is the parallelism relation of $\mathrm{AG}(3,q)$. In the following section we choose first as Ω the tangent lines of a conic in E, and as G the group $\mathrm{PGO}(3,q)$. Then we take as Ω the tangent lines of a unital in E and as G the group $\mathrm{PGU}(3,q)$. Thus, we obtain the following imprimitive permutation groups, $(T \cdot \mathrm{GO}(3,q), \Pi(\Omega), R)$ and $(T \cdot \mathrm{GU}(3,q), \Pi(\Omega), R)$. In this way we construct divisible designs admitting, as an automorphism group, an orthogonal group and a unitary group, respectively. Moreover, in the special case of the conic when q is odd we construct also 3-divisible designs admitting, as an automorphism group, an orthogonal group. Notice that a divisible design is connected with a constant weight code and that the automorphism group of the design induces faithfully an automorphism group of the code, cf. [Sch, SchSp3]. Hence our results can be translated into those on constant weight codes.

2 Preliminary definitions and results

Let X be a set of size v and R an equivalence relation on X. We denote by $[x]$, with $x \in X$, the R-equivalence class containing x and by $\mathcal{R}$ the set of all R-equivalence classes.

2.1 Definition *A subset Y of X is called R-transversal if $\mid Y \cap [x] \mid \leq 1$ for all $x \in X$.*

Consider a family $\mathcal{B}$ of *R-transversal* subsets of X, and let t, s, k, λ be positive integers with $t \leq k < v$. Then, one has the following definition.

2.2 Definition *A triple $D = (X, \mathcal{B}, \mathcal{R})$ is called t-(s, k, λ_t)-divisible design (or shorter t-divisible design) if*

(i) $\mid [x] \mid = s$ *for all* $x \in X$*;*

(ii) $\mid B \mid = k$ *for all* $B \in \mathcal{B}$*;*

(iii) for every R-transversal t-subset Y of X, there exist exactly λ_t elements of $\mathcal{B}$ containing Y.

The elements of X are called points, those of $\mathcal{B}$ blocks, and the elements of $\mathcal{R}$ point classes.

One can show that, for $t \geq 3$, a t-divisible design is as well a $(t-1)$-divisible design with

$$\lambda_{t-1} = \lambda_t(v - st + s)(k - t + 1)^{-1}.$$

Notice that, for $s = 1$, a t-divisible design is a t-design. If $t = 2$ we call a t-divisible design just divisible design. If D is a (s, k, λ_2)-divisible design, then the number of blocks through a given point is a constant number r and

$$r(k-1) = \lambda_2(v - s) \quad \text{and} \quad bk = vr,$$

where b denotes the number of blocks of D. For examples and background on divisible designs see for instance [BJL] or [Ha].

2.3 Definition *Let G be a group operating on the set X and R an equivalence relation on X which is G-admissible, that is*

$$x, y \in X \textit{ and } xRy \textit{ imply } x^g R y^g \textit{ for all } g \in G,$$

then the triple $\Lambda = (G, X, R)$ is called an R-permutation group (see [Sp1]*). The group G induces a permutation group on X, but not necessarily faithfully.*

An $R-$permutation group Λ is called *t-R-homogeneous* (and *t-R-transitive* respectively) if for every two R-transversal subsets of X of size t, $S = \{x_1, ..., x_t\}$ and $S' = \{y_1, ..., y_t\}$, there exists an element $g \in G$ such that $S' = S^g$ (and $y_i = {x_i}^g$ for every $i = 1, 2, ..., t$ respectively). Notice that, if G is *t-R-transitive* , then G is *t-R-homogeneous.*
t-divisible designs are closely connected to R-permutation groups as the following proposition shows.

2.4 Proposition.[Sp1] *Let $\Lambda = (G, X, R)$ be a finite t-R-transitive R-permutation group, and let B be an R-transversal subset of X of size k, with $t \leq k < v := \mid X \mid$. Then, the incidence structure $D(\Lambda, B) = (X, B^G, \mathcal{R})$, where $B^G = \{B^g \mid g \in G\}$, is a t-divisible design with the following parameters, $b = \frac{|G|}{|G_B|}$, $s = \mid [x] \mid$ for some $x \in X$, $k = \mid B \mid$ and $\lambda_t = \mid G \mid \binom{k}{t} (\mid G_B \mid \binom{v/s}{t} s^t)^{-1}$; here G_B denotes the setwise stabilizer of B. Moreover, $D(\Lambda, B)$ admits G as a point- and a block-transitive automorphism group.*

3 General results

The following general theorem helps to find 2-R-transitive R-permutation groups. Consider the 3-dimensional projective space $\mathrm{PG}(3, q)$ over $K = \mathrm{GF}(q)$, with $q = p^e$, p a prime, and fix, as improper plane, the plane $E := \{X_3 = 0\}$. If G is a subgroup of the collineation group $\mathrm{PGL}(3, q)$ of E, then, $G \cong \overline{G}/\overline{G} \cap Z(\mathrm{GL}(3, q))$, where $\overline{G}$ is a suitable subgroup of $\mathrm{GL}(3, q)$ and $Z(\mathrm{GL}(3, q))$ is the centre of $\mathrm{GL}(3, q)$.
We embed $\overline{G}$ in $\mathrm{PGL}(4, q)$ as follows,

$$\widetilde{G} = \left\{ \begin{pmatrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid A \in \overline{G} \right\};$$

notice that $\widetilde{G} \cap Z(\mathrm{GL}(4, q)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. So, $\widetilde{G}/\widetilde{G} \cap Z(\mathrm{GL}(4, q))$ is isomorphic to $\widetilde{G}$. Define $\mathrm{AG}(3, q)$ as the affine space with ideal plane E and T as its translation group.

3.1 Proposition. *Let Ω be a set of lines of E, $\mid \Omega \mid \geq 2$, and let $\Pi(\Omega)$ be the following set,*

$$\Pi(\Omega) = \{\pi \text{ a plane of } \mathrm{AG}(3, q) \mid \pi \cap E \in \Omega\}.$$

- (i) *If G is a subgroup of $\mathrm{PGL}(3, q)$ that acts 2-transitively on the set Ω then the group $(T \cdot \widetilde{G}, \Pi(\Omega), R)$ is a 2-R-transitive R-permutation group;*
- (ii) *if G is a subgroup of $\mathrm{PGL}(3, q)$ which acts 3-transitively on the set Ω. Then, if there do not exist three concurrent lines in Ω, the group $(T \cdot \widetilde{G}, \Pi(\Omega), R)$ is a 3-R-transitive R-permutation group;*

here T and $\widetilde{G}$ are defined as above and R denotes the parallelism relation.

Proof (i) Notice that the group $\widetilde{G}$ is still 2-transitive on Ω and respects the parallelism relation; let $\{\pi_1, \pi_2\}$ and $\{\pi_1', \pi_2'\}$ be two R-transversal 2-subsets of $\Pi(\Omega)$. Then, $\pi_1 \cap \pi_2 = \ell$ and $\pi_1' \cap \pi_2' = \widetilde{\ell}$, where $\widetilde{\ell}$ and ℓ are two affine lines. So, if we choose a point P on ℓ and a point P' on $\widetilde{\ell}$, there exists a τ in T such that $P^\tau = P'$. Therefore, we can suppose that $\{\pi_1, \pi_2, \pi_1', \pi_2'\}$ intersect in $O = (0, 0, 0, 1)K$. Moreover, $(T \cdot \widetilde{G})_O = (\widetilde{G})_O = \widetilde{G}$ and $\widetilde{G}$ is 2-transitive on Ω. So, since through a point and a line exists a unique plane, the assertion follows. (ii) Let $\{\pi_1, \pi_2, \pi_3\}$ and $\{\pi_1', \pi_2', \pi_3'\}$ be two R-transversal 3-subsets of $\Pi(\Omega)$; notice that, since $\pi_1 \cap E$, $\pi_2 \cap E$, $\pi_3 \cap E$ and $\pi_1' \cap E$, $\pi_2' \cap E$, $\pi_3' \cap E$ respectively form a triangle, then $\pi_1 \cap \pi_2 \cap \pi_3 = \{P\}$, where P is an affine point. In fact, if $\pi_1 \cap \pi_2 = \{\ell\}$ and π_3 does not intersect with π_1 and π_2, then $\pi_3 \cap E$ will be concurrent with $\pi_1 \cap E$ and $\pi_2 \cap E$. So, we can suppose that π_1 , π_2 , π_3, π_1' , π_2' , π_3' intersect in $O = (0,0,0,1)K$. Then, $(T \cdot \widetilde{G})_O = \widetilde{G}_O = \widetilde{G}$ and an element $g \in \widetilde{G}$ maps $\pi_1 \cap E$, $\pi_2 \cap E$, $\pi_3 \cap E$ onto $\pi_1' \cap E$, $\pi_2' \cap E$, $\pi_3' \cap E$. So the assertion follows. $\square$

3.2 Remarks.

(i) Let $(T \cdot \widetilde{G}, \Pi(\Omega), R)$ be the R-permutation group defined above. We show that $T \cdot \widetilde{G}$ acts faithfully on $\Pi(\Omega)$ if not all the lines of Ω do intersect in a point, and acts not faithfully if all the lines of Ω are concurrent. Suppose that not all the lines of Ω do intersect in a point, then every affine point is the intersection of three suitable planes of $\Pi(\Omega)$. Hence $T \cdot \widetilde{G}$ acts faithfully on $\Pi(\Omega)$. Now, suppose that all the lines of Ω intersect in a point. Then the sets of not parallel affine planes through the lines of Ω intersect in an affine line, and these affine lines are parallel. So, if τ is a translation in the direction of these lines, then $\pi^\tau = \pi$ for all $\pi \in \Pi(\Omega)$, and $\tau \neq id$. Therefore, at least τ is in the kernel of the action.

(ii) Let (H, X, R) be an t-R-transitive R-permutation group, and let $\phi : H \longrightarrow \widetilde{H}$ with $\widetilde{H} \subseteq \mathrm{Sym}(X)$ be the homomorphism of the action on X. Moreover, suppose that $\mathrm{Ker}\phi \neq id$ and suppose that, using the Proposition 2.4, we have constructed a t-(s,k,λ) divisible design $\mathcal{D} = (X, B^H, \mathcal{R})$, with b blocks. So, we have

$$b = \mid H \mid \cdot \mid H_B \mid^{-1} = (\mid \widetilde{H} \mid \cdot \mid \mathrm{Ker}\phi \mid)(\mid \widetilde{H_B} \mid \cdot \mid \mathrm{Ker}\phi \mid)^{-1} = \mid \widetilde{H} \mid \cdot \mid \widetilde{H_B} \mid^{-1},$$

where $\widetilde{H} \cong H/\mathrm{Ker}\phi$. So, we can use alternatively H or $H/\mathrm{Ker}\phi$ to determine the parameters of a divisible design. Notice that the automorphism group of $\mathcal{D}$ contains $\widetilde{H}$

3.3 Remark.

Every t-(s,k,λ_t)-divisible design $\mathcal{D}$ with n point classes corresponds to a code $\mathcal{C}$ having the following properties; (i) $\mathcal{C} \subseteq (\mathbb{Z}_{s+1})^n$, (ii) $\mathcal{C}$ has constant weight k and (iii) For any t different positions $i_1, \ldots, i_t$ and for any elements $j_1, \ldots, j_t \in \mathbb{Z}_{s+1} \setminus \{0\}$ there exist exactly λ_t words of $\mathcal{C}$ with coordinate j_u at position i_u for $u = 1, 2, \ldots, t$.
Since $\mathrm{Aut}\mathcal{D}$ induces faithfully a subgroup of $\mathrm{Aut}\,\mathcal{C}$ (cf. [Sch, SchSp3]) the methods used here allow as well to construct constant weight codes with large automorphism group.

4 The construction

A) The conic

Consider $\mathrm{PG}(2,q)$, $q = p^e$, consider the non singular conic $\widetilde{\mathcal{C}}$ with equation $X_1^2 - X_0X_2 = 0$, and the group $G = \mathrm{PGO}(3,q)$ operating on it. We have the following theorem, see for instance p. 143 of [Hr], p.53 of [HP] or p.120 of [Hu].

4.1 Theorem. $\mathrm{PGO}(3,q)$ *on* $\widetilde{\mathcal{C}}$ *is permutation isomorphic to* $\mathrm{PGL}(2,q)$ *on* $\mathrm{PG}(1,q)$.

The isomorphism from $\mathrm{PGL}(2,q)$ onto $\mathrm{PGO}(3,q)$ is induced by

$$\sigma : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}, \quad ad-bc \neq 0.$$

But notice that these matrices are in general not elements of $\mathrm{GO}(3,q)$. An immediate corollary of this theorem is the following,

4.2 Corollary $\mathrm{PGO}(3,q)$ *acts sharply* 3*-transitively on the conic* $\widetilde{\mathcal{C}}$ *with equation* $X_1^2 - X_0X_2 = 0$.

Notice that, since $\mathrm{PGO}(3,q)$ acts 3-transitively on the points of $\widetilde{\mathcal{C}}$, it acts 3-transitively also on the tangent lines of the conic $\widetilde{\mathcal{C}}$. Consider $\mathrm{PG}(3,q)$ and embed the conic $\widetilde{\mathcal{C}}$ as follows,

$$\mathcal{C} : \begin{cases} X_1^2 - X_0X_2 & = & 0 \\ X_3 & = & 0. \end{cases}$$

Then the group $\widetilde{G} = \left\{ \begin{pmatrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid A \in \mathrm{GO}(3,q) \right\}$ is isomorphic to $\mathrm{GO}(3,q)$ and acts 3-transitively on the tangent lines of $\mathcal{C}$. Therefore, by Proposition 3.1, the R-permutation group $(T \cdot \widetilde{G}, \Pi(\Omega), R)$, where Ω is the set of tangent lines of $\mathcal{C}$, is at least 2-R-transitive.

4.3 Theorem. *Let* $q = p^e$*, where* p *is a prime and let* i *be an integer such that* $i \mid e$.

(*i*) *if* q *is odd, there exists a* $2\text{-}(q, p^i+1, \frac{q(q-1)}{p^i-1})$*-divisible design* $\mathcal{D}$ *with* q^2+q *points having as a point- and a block-transitive automorphism group* $T \cdot \widetilde{G}$ *with* $\widetilde{G} \cong \mathrm{GO}(3,q)$;

(*ii*) *if* q *is even, there exists a* $2\text{-}(q, p^i+1, \frac{q-1}{p^i-1})$*-divisible design with* q^2+q *points having, as a point- and a block-transitive automorphism group* $\widetilde{G} \cdot (T/\mathrm{Ker}\phi)$*, with* $\widetilde{G} \cong \mathrm{GO}(3,q)$*, where* $\mathrm{Ker}\phi$ *is the kernel of the action on* $\Pi(\Omega)$*, with* Ω *the set of tangent lines of* $\mathcal{C}$.

Moreover, if q *is odd the design* $\mathcal{D}$ *constructed here is in fact a* $3\text{-}(q, p^i+1, 1)$*-divisible design with a point- and a block- transitive automorphism group.*

Proof (*i*) Let q be odd, let $\mathrm{PG}(3,q)$ be the projective space over $K = \mathrm{GF}(q)$, let $\mathcal{C}$ be the conic defined above and let Ω be the set of tanget lines of $\mathcal{C}$. Consider the R-permutation group $(T \cdot \widetilde{G}, \Pi(\Omega), R)$ and notice that, by (*ii*) of Proposition 3.1 and by (*i*) of Remarks 3.2, it is 3-R-transitive and its action is faithfull. We construct a 3-divisible design and hence a divisible design using Proposition 2.4. Therefore, we choose an R-transversal subset of $\Pi(\Omega)$, namely B, as follows;

$$B := \Pi_O(\Omega') = \{\pi \text{ a plane of } \mathrm{AG}(3,q) \mid \pi \cap E \in \Omega' \text{ and } O = (0,0,0,1)K \in \pi\},$$

where

$$\Omega' = \left\{ \ell \in \Omega \mid \text{ tangent point of } \ell \text{ is } (1,0,0,0)K \text{ or } (t^2,t,1,0)K,\ t \in \mathrm{GF}(p^i) \right\}.$$

Now, we compute the stabilizer $(T\cdot\widetilde{G})_B$ of B. Notice that $(T\cdot\widetilde{G})_B=\widetilde{G}_{\Omega'}$, since T is regular on the affine points, $\cap\{\pi \mid \pi\in B\}=O$ and $\widetilde{G}$ fixes $O=(0,0,0,1)K$. Furthermore, $\widetilde{G}\cong \mathrm{GO}(3,q)$ and $\mathrm{PGO}(3,q)_{\Omega'}\cong(\mathrm{GO}(3,q)/Z(\mathrm{GO}(3,q)))_{\Omega'}\cong \mathrm{GO}(3,q)_{\Omega'}/Z(\mathrm{GO}(3,q))$. Therefore, it is sufficient to compute $\mid \mathrm{PGO}(3,q)_{\Omega'}\mid$. Moreover, $\mid \mathrm{PGO}(3,q)_{\Omega'}\mid=\mid \mathrm{PGO}(3,q)_{\mathcal{C}'}\mid$, where $\mathcal{C}'$ is the set of tangent points of the lines of Ω'. If $g\in \mathrm{PGO}(3,q)$ fixes $\mathcal{C}'$, then g maps three points P_1, P_2, P_3 of $\mathcal{C}'$ on to three points P_1', P_2', P_3' of $\mathcal{C}'$. Furthermore, there exists an element $g'\in \mathrm{PGO}(3,p^i)$ doing this and fixing $\mathcal{C}'$. Therefore, $g=g'$ since $\mathrm{PGO}(3,q)$ is 3-regular. Then, $\mid(T\cdot\widetilde{G})_B\mid=\mid \mathrm{PGO}(3,p^i)\mid\cdot\mid Z(\mathrm{GO}(3,q))\mid=p^i(p^{2i}-1)2$. Therefore, by Proposition 2.4, the parameters of the 3-divisible design, $\mathcal{D}=(\Pi(\Omega),B^{T\cdot\widetilde{G}},\mathcal{R})$, with $\widetilde{G}\cong \mathrm{GO}(3,q)$, are the following

$$v=\mid\Pi(\Omega)\mid=q(q+1),\quad s=q,\quad k=\mid B\mid=p^i+1 \quad\text{and}$$

$$\lambda_3=\frac{|T||\widetilde{G}|}{|(T\widetilde{G})_B|}\frac{\binom{k}{3}}{\binom{v/s}{3}s^3}=\frac{q^3q(q^2-1)2}{p^i(p^{2i}-1)2}\frac{\binom{p^i+1}{3}}{\binom{q(q+1)/q}{3}q^3}=1$$

And the parameters of $\mathcal{D}$ as 2-divisible design are

$$v=q(q+1),\ s=q,\ k=p^i+1 \text{ and } \lambda_2=\frac{(v-2s)\lambda_3}{k-2}=\frac{q(q-1)}{p^i-1}.$$

Notice that an automorphism group of these divisible designs is $T\cdot\widetilde{G}$, with $\widetilde{G}\cong \mathrm{GO}(3,q)$, and the order of this group is $2(q^2-1)q^4$. (ii) As in the case (i) we consider the conic $\mathcal{C}$ and the now 2 -R - transitive R-permutation group $(T\cdot\widetilde{G},\Pi(\Omega),R)$. It is known that the lines of Ω intersect in the nucleus of the conic, $N=(0,1,0,0)K$, so by (i) of Remarks 3.2, the action of this group is not faithfull. We want to find the kernel of the action. Notice that $\widetilde{G}\cong \mathrm{GO}(3,q)$ acts faithfully on $\Pi(\Omega)$. In fact, suppose that there exists $\widetilde{g}\in\widetilde{G}$ fixing the planes of $\Pi(\Omega)$, then $\widetilde{g}$ induces an homology, $\overline{g}$, in E with centre N. But $\overline{g}$ has to fix all the points of the conic, since it fixes the tangent lines of the conic. Therefore, it is the identity. Now, let $\tau\in T$ and $g\in\widetilde{G}$, with $\tau g\in \mathrm{Ker}\phi$; then τ fixes (as g) the line ON; hence, by Remarks 3.2, $\tau\in\mathrm{Ker}\phi$ and therefore $g=id$. So, only the translations in direction of the line ON form the kernel. Also in this case, we construct a divisible design using Proposition 2.4. We choose the same R-transversal subset as in case (i), $B=\Pi_O(\Omega')$, and we compute the stabilizer. Then, $\mid \mathrm{PGO}(3,q)_{\Omega'}\mid$ is the same as in the case (i). Therefore, the 2-divisible design $(\Pi(\Omega),B^{T\cdot\widetilde{G}},\mathcal{R})$ has got the following parameters;

$v=q(q+1),\ s=q,\ k=p^i+1$ and finally, by Remark 3.2 (ii),

$\lambda_2=\frac{|T\cdot\widetilde{G}|/|\mathrm{Ker}\phi|}{|(T\cdot\widetilde{G})_B|/|\mathrm{Ker}\phi|}\frac{k(k-1)}{v(v-s)}=\frac{q^3q(q^2-1)}{qp^i(p^{2i}-1)}\frac{p^i(p^i+1)}{q(q+1)(q^2+q-q)}=\frac{q-1}{p^i-1}$, and has, as a point- and block-transitive automorphism group, the group $\widetilde{G}\cdot(T/\mathrm{Ker}\phi)$, with $\widetilde{G}\cong \mathrm{GO}(3,q)$, of order $q^3(q^2-1)$.$\square$.

4.4 Theorem *Let $q=p^e$, with p an odd prime. Then, there exist a* 2-$(q,2,1)$-*divisible design and a* 3-$(q,3,1)$-*divisible design with $q(q+1)$ points each. Both they admit as a point- and block-transitive automorphism group, $T\widetilde{G}$ with $\widetilde{G}\cong \mathrm{GO}(3,q)$.*

Proof Let $(T\cdot\widetilde{G},\Pi(\Omega),R)$, where $\widetilde{G}\cong \mathrm{GO}(3,q)$ and Ω is the set of tangent lines of $\mathcal{C}$, be the 3-R-transitive permutation group defined above. Again, we use the Proposition 2.4 to construct a divisible design. In particular, if ℓ_1 and ℓ_2 are two fixed tangent lines of $\mathcal{C}$ we choose the R-transversal subset $B:=\{\pi_1,\pi_2\}$ of $\Pi(\Omega)$ where π_1,π_2 are the two planes of $\mathrm{AG}(3,q)$ trough $O=(0,0,0,1)K$ with $\pi_1\cap E=\ell_1$ and $\pi_2\cap E=\ell_2$. We compute

$(T\cdot \widetilde{G})_B$. Notice that, if $\{P\} = \ell_1 \cap \ell_2$, the translations in the direction of the affine line trough O and P fix B. We denote this subgroup of T by $\widetilde{T}$. Then, $\widetilde{T} \subseteq (T\cdot \widetilde{G})_B$, and $\mid \widetilde{T} \mid= q$. Moreover, the group $\mathrm{GO}(3,q)_{\{\ell_1,\ell_2\}}$ fixes B; note that $\mathrm{PGO}(3,q)_{\{\ell_1,\ell_2\}} = \mathrm{GO}(3,q)_{\{\ell_1,\ell_2\}}/Z(\mathrm{GO}(3,q))$ and $\mathrm{PGO}(3,q)_{\{\ell_1,\ell_2\}} = \mathrm{PGO}(3,q)_P$. In general, $\mathrm{PGO}(3,q)$ acts transitively on the external points of a conic $X_1^2 - X_0X_2 = 0$ in $\mathrm{PG}(2,q)$; in fact the length of the orbit of the point $(0,1,0)K$ is $\frac{q(q+1)}{2}$. Therefore, $\mid \mathrm{PGO}(3,q)_P \mid= 2(q-1)$, and $\mid \mathrm{GO}(3,q)_{\{\ell_1,\ell_2\}} \mid= 2(q-1) \mid Z(\mathrm{GO}(3,q)) \mid= 2(q-1)2$. Then, $\widetilde{T}\cdot\mathrm{GO}(3,q)_{\{\ell_1,\ell_2\}} \subseteq (T\cdot \widetilde{G})_B$. Now we show that $\widetilde{T}\cdot\mathrm{GO}(3,q)_{\{\ell_1,\ell_2\}} = (T\cdot \widetilde{G})_B$. Let $\tau g \in (T\cdot \widetilde{G})_B$, thus ${\pi_1}^{\tau g} \in B$, and ${\pi_2}^{\tau g} \in B$. Suppose that ${\pi_1}^{\tau g} \neq \pi_1$, then ${\pi_1}^{\tau g} = {\pi_1}^{g\tau_1} = \pi_2$ and ${\pi_2}^{\tau g} = {\pi_2}^{g\tau_1} = \pi_1$. Now ${\pi_1}^g$ is a plane through O, through P, and through ℓ_2, since the translations fix E pointwise, hence it is π_2. Similarly ${\pi_2}^g = \pi_1$. Therefore, $\tau_1 \in \widetilde{T}$, and $g \in \mathrm{GO}(3,q)_{\{\ell_1,\ell_2\}}$. Now, suppose that ${\pi_1}^{\tau g} = \pi_1$, and consequently ${\pi_2}^{\tau g} = \pi_2$. Then, ${\pi_1}^{\tau g} = {\pi_1}^{g\tau_1} = \pi_1$. Therefore, ${\pi_1}^g = \pi_1$, similarly ${\pi_2}^g = \pi_2$ and consequently $g \in \mathrm{GO}(3,q)_{\{\ell_1,\ell_2\}}$ and $\tau_1 \in \widetilde{T}$. So, we obtain a divisible design with the following parameters,

$v = q(q+1)$, $s = q$, $k = 2$ and

$$\lambda_2 = \frac{|T\cdot\mathrm{GO}(3,q)|}{|(T\cdot\mathrm{GO}(3,q))_B|}\frac{k(k-1)}{v(v-s)} = \frac{q^3q(q^2-1)2}{q2(q-1)2}\frac{2}{q(q+1)[q(q+1)-q]} = 1.$$

An automorphism group of this divisible design is $T\cdot\widetilde{G}$ with $\widetilde{G} \cong \mathrm{GO}(3,q)$.
Again, consider the 3-R-transitive R-permutation group $(T\cdot\widetilde{G}, \Pi(\Omega), R)$. Let ℓ_1, ℓ_2, ℓ_3 be three fixed tangent lines of $\mathcal{C}$. Choose, as an R-transversal subset of $\Pi(\Omega)$, $B := \{\pi_1,\pi_2,\pi_3\}$, where π_1, π_2, π_3 are three planes of $\mathrm{AG}(3,q)$ through $O = (0,0,0,1)K$ with $\pi_1 \cap E = \ell_1$, $\pi_2 \cap E = \ell_2$ and $\pi_3 \cap E = \ell_3$. We compute $(T\cdot\widetilde{G})_B$. Notice that $(T\cdot\widetilde{G})_B = \widetilde{G}_B = \widetilde{G}_{\{\pi_1,\pi_2,\pi_3\}} = \widetilde{G}_{\{\ell_1,\ell_2,\ell_3\}}$ and $\mathrm{PGO}(3,q)_{\{\ell_1,\ell_2,\ell_3\}} \cong \mathrm{GO}(3,q)_{\{\ell_1,\ell_2,\ell_3\}}/Z(\mathrm{GO}(3,q))$. If P_1, P_2, P_3 are the tangent points of ℓ_1, ℓ_2, ℓ_3, respectively then $\mathrm{PGO}(3,q)_{\{\ell_1,\ell_2,\ell_3\}} = \mathrm{PGO}(3,q)_{\{P_1,P_2,P_3\}}$. Moreover, $\mathrm{PGO}(3,q)$ acts sharply 3-transitively on the points of the conic, so the pointwise stabilizer $\mathrm{PGO}(3,q)_{P_1,P_2,P_3} = id$. Therefore, only the permutations on three elements are in $\mathrm{PGO}(3,q)_{\{P_1,P_2,P_3\}}$. So $\mathrm{PGO}(3,q)_{\{P_1,P_2,P_3\}} \cong \mathrm{S}_3$ and $\mid (T\cdot\widetilde{G})_B \mid= \mid \mathrm{S}_3 \mid\mid Z(\mathrm{GO}(3,q)) \mid= 12$. Hence, the 3-divisible design $(\Pi(\Omega), B^{T\cdot\widetilde{G}}, \mathcal{R})$ has the following parameters

$v = q(q+1)$, $s = q$, $k = 3$ and

$$\lambda_3 = \frac{|T\cdot\mathrm{GO}(3,q)|}{|(T\cdot\mathrm{GO}(3,q))_B|}\frac{\binom{k}{3}}{\binom{v/s}{3}s^3} = \frac{q^3q(q^2-1)2}{12}\frac{1}{q^3q(q+1)(q-1)/3!} = 1 \text{ and } \lambda_2 = q(q-1).$$

The constructed 3-divisible design admits $T\cdot\widetilde{G}$, with $\widetilde{G} \cong \mathrm{GO}(3,q)$ as a point- and block-transitive automorphism group.□
Notice that the constructed 2-divisible design is a complete multipartite graph on $(q+1)\times q$ points.

B) The unital

Let $\mathrm{PG}(3,q^2)$, $q = p^e$, be the projective space over $K = \mathrm{GF}(q^2)$ and let E be the improper plane defined above. Then, E is a $\mathrm{PG}(2,q^2)$. Consider, in $\mathrm{PG}(3,q^2)$, the following unital (Hermitian curve), see [Hr] p.146;

$$\mathcal{U} : \begin{cases} X_0^{q+1} + X_1^{q+1} + X_2^{q+1} & = & 0 \\ X_3 & = & 0. \end{cases}$$

The group $\mathrm{PGU}(3,q^2)$ acts on $\mathrm{PG}(2,q^2)$ fixing the set of points with equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$ and is 2-transitive on these points, (see [Lü] p. 156). Now, we embed $\mathrm{GU}(3,q^2)$ in the collineation group $\mathrm{PGL}(4,q^2)$ of $\mathrm{PG}(3,q^2)$, as follows,

$$\widetilde{G} = \left\{ \begin{pmatrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid A \in \mathrm{GU}(3,q^2) \right\}.$$

The group $\widetilde{G}$ is isomorphic to $\mathrm{GU}(3,q^2)$ and fixes the plane E. It is known that a unital is a 2-design with q^3+1 points and that, through every point of it, there exist q^2 secant lines and 1 tangent line, (see [Hr] p.147). So, we consider the set Ω of the q^3+1 tangent lines of $\mathcal{U}$. By Proposition 3.1, the R-permutation group $(T \cdot \widetilde{G}, \Pi(\Omega), R)$ is 2-R-transitive. Notice that the tangent lines of $\mathcal{U}$ do not all intersect in a point (see [Hr] p.147), so, by (i) of Remarks 3.2, the action of this group is faithfull.

4.5 Theorem. *Let $q = p^e$, where p is a prime. Then there exist $(q^2, q^3, q^2(q^3-1))$-, $(q^2, q+1, 1)$- and $(q^2, 2, 1)$-divisible designs with $q^2(q^3+1)$ points each admitting $T \cdot \widetilde{G}$ with $\widetilde{G} \cong \mathrm{GU}(3,q^2)$ as a point- and block-transitive automorphism group; moreover if q is odd there exists a $(q^2, q+1, q^2(q+1))$-divisible design with $q^2(q^3+1)$ points and admitting $T \cdot \widetilde{G}$ with $\widetilde{G} \cong \mathrm{GU}(3,q^2)$ as a point- and block-transitive automorphism group.*

Proof (i) Consider $\mathrm{PG}(3,q^2)$, the projective space over $K = \mathrm{GF}(q^2)$, the unital $\mathcal{U}$ and the 2-R-transitive R-permutation group $(T \cdot \widetilde{G}, \Pi(\Omega), R)$, where $\widetilde{G} \cong \mathrm{GU}(3,q^2)$ and Ω is the set of tanget lines of $\mathcal{U}$. Here again, we can construct a divisible design using Proposition 2.4. Let Ω' be a subset of Ω as defined by $\Omega' = \{\ell \in \Omega \mid \text{the tangent point of } \ell \text{ is in } \mathcal{U} - \{P\}\}$, where P is a fixed point of $\mathcal{U}$.
Let $B = \Pi_O(\Omega') = \{\pi \text{ a plane of } \mathrm{AG}(3,q^2) \mid O = (0,0,0,1)K \in \pi \text{ and } \pi \cap E \in \Omega'\}$; it is an R-transversal subset of $\Pi(\Omega)$. We compute the stabilizer of B. Then $(T \cdot \widetilde{G})_B = \widetilde{G}_B = \widetilde{G}_{\Omega'}$ since $\widetilde{G}$ fixes O and since $\{\mathrm{O}\}$ is the intersection of the planes in B. Moreover, $\widetilde{G}_{\Omega'} \cong \mathrm{GU}(3,q^2)_{\Omega'}$, and $\mathrm{GU}(3,q^2)_{\Omega'} = \mathrm{GU}(3,q^2)_{\mathcal{U}-\{P\}} = \mathrm{GU}(3,q^2)_P$. Therefore,

$$\mid (T \cdot \widetilde{G})_B \mid = \mid \mathrm{GU}(3,q^2)_P \mid = q^3(q^2-1)(q+1).$$

So, the parameters of the divisible design $(\Pi(\Omega), B^{T \cdot \widetilde{G}}, \mathcal{R})$, with $\widetilde{G} \cong \mathrm{GU}(3,q^2)$, are the following;

$$v = \mid \Pi(\Omega) \mid = q^2(q^3+1), \;\; s = q^2, \;\; k = q^3 \;\; \text{and}$$

$$\lambda_2 = \frac{|T \cdot \mathrm{GU}(3,q^2)|}{|\mathrm{GU}(3,q^2)_P|} \frac{k(k-1)}{v(v-s)} = \frac{q^6(q^3+1)q^3(q^2-1)(q+1)}{q^3(q^2-1)(q+1)} \frac{q^3(q^3-1)}{q^2(q^3+1)[q^2(q^3+1)-q^2]} = q^2(q^3-1).$$

(ii) Now, we take the same R-permutation group as above and choose a secant h of $\mathcal{U}$. So $\mid h \cap \mathcal{U} \mid = q+1$. Consider the $q+1$ tangent lines having as contact points the points of h, they intersect in the polar point S of h (see [Hr] p.147, p.148). Let $(T \cdot \widetilde{G}, \Pi(\Omega), R)$, with

$\widetilde{G} \cong \mathrm{GU}(3,q^2)$ be the 2-R-transitive R-permutation group used above. We choose $\widetilde{B}$, an R-transversal subset of $\Pi(\Omega)$, as follows;

$$\widetilde{B} := \Pi_O(\widetilde{\Omega}) = \Big\{ \pi \text{ a plane of } \mathrm{AG}(3,q^2) \ \mid \ \pi \cap E \in \widetilde{\Omega} \text{ and } O = (0,0,0,1)K \in \pi \Big\},$$

where

$$\widetilde{\Omega} = \{\ell \in \Omega \mid \text{ the tangent point of } \ell \text{ is a point of } h\} = \{\ell \in \Omega \ \mid \ S \in \ell\}.$$

We compute the stabilizer of $\widetilde{B}$. Notice that the group of the translation in the direction of the line OS, $\widetilde{T}$, is contained in $(T \cdot \widetilde{G})_B$. Also $\widetilde{G}_h$, is contained in $(T \cdot \widetilde{G})_{\widetilde{B}}$, since $\widetilde{G}$ fixex O. Thus, $\widetilde{T} \cdot \widetilde{G}_h \subseteq (T \cdot \widetilde{G})_{\widetilde{B}}$. We prove that $\widetilde{T} \cdot \widetilde{G}_h$ is exactly the stabilizer. In fact, suppose that there exists an element $r \in (T \cdot \widetilde{G})_B$, $r = \tau g = g\tau_1$, with $\tau_1 \in T$ and $g \in \widetilde{G}$. Let $\pi_1, \pi_2 \in \widetilde{B}$ with $\pi_1 \neq \pi_2$ and $\pi_i' := \pi_i^{\tau g}$. Then $(OS)^{\tau g} = (\pi_1 \cap \pi_2)^{\tau g} = \pi_1' \cap \pi_2' = OS$ and $S^{\tau g} = S$. Thus $g \in \widetilde{G}_h$ and $\tau_1 \in \widetilde{T}$. Notice that $\widetilde{G} \cong \mathrm{GU}(3,q^2)$ and $\mathrm{PGU}(3,q^2)_h \cong \mathrm{GU}(3,q^2)_h/Z(\mathrm{GU}(3,q^2))$. Therefore, it sufficient to compute $\mid \mathrm{PGU}(3,q^2)_h \mid$. Since $\mathrm{PGU}(3,q^2)$ acts 2-transitively on the points of $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$, it acts transitively on the secant lines. Moreover, the number of secant lines is $q^2(q^2-q+1)$. Thus, $\mid \mathrm{PGU}(3,q^2)_h \mid= q(q^2-1)(q+1)$, and $\mid \widetilde{T} \mid\mid \mathrm{GU}(3,q^2)_h \mid= q^2q(q^2-1)(q+1)(q+1)$. Therefore, the divisible design has the following parameters

$$v = q^2(q^3+1), \ s = q^2, \ k = q+1 \text{ and}$$

$$\lambda_2 = \tfrac{|T \cdot \mathrm{GU}(3,q^2)|}{|(T \cdot \mathrm{GU}(3,q^2)_B|} \tfrac{k(k-1)}{v(v-s)} = \tfrac{q^6(q^3+1)q^3(q^2-1)(q+1)}{q^2q(q^2-1)(q+1)(q+1)} \tfrac{q(q+1)}{q^2(q^3+1)[q^2(q^3+1)-q^2]} = 1.$$

The constructed divisible design admits $T \cdot \widetilde{G}$ with $\widetilde{G} \cong \mathrm{GU}(3,q^2)$ as a point- and block-transitive automorphism group.

(iii) Again, let $(T \cdot \widetilde{G}, \Pi(\Omega), R)$, with $\widetilde{G} \cong \mathrm{GU}(3,q^2)$ be the 2-R-transitive R-permutation group used above. Let ℓ_1, ℓ_2 be two fixed tangent lines of $\mathcal{U}$, and consider the R-transversal subset $\hat{B} = \{\pi_1, \pi_2\}$ of $\Pi(\Omega)$, where π_1, π_2 are the two planes of $AG(3,q^2)$ through $O = (0,0,0,1)K$ and such that $\pi_1 \cap E = \ell_1$, $\pi_2 \cap E = \ell_2$. We want to compute $\mid (T \cdot \widetilde{G})_{\hat{B}} \mid$. Notice that, if $\ell_1 \cap \ell_2 = \{S\}$, then $\widetilde{T}$, the group of the translations in the direction of the line OS, is contained in $(T \cdot \widetilde{G})_{\hat{B}}$. Moreover, $\widetilde{G}_{\hat{B}} = \widetilde{G}_{\{\ell_1,\ell_2\}} = \widetilde{G}_{\{P,Q\}}$, where $P = \ell_1 \cap \mathcal{U}$ and $Q = \ell_2 \cap \mathcal{U}$, and $\mathrm{PGU}(3,q^2)_{\{P,Q\}} \cong \mathrm{GU}(3,q^2)_{\{P,Q\}}/Z(\mathrm{GU}(3,q^2))$. So, it is sufficient to compute $\mid \mathrm{PGU}(3,q^2)_{\{P,Q\}} \mid$. It is known (see [Lü] pg. 156) that the pointwise stabilizer of P, Q, $\mathrm{PGU}(3,q^2)_{P,Q}$ is a cyclic group of order q^2-1. Moreover, since $\mathrm{PGU}(3,q^2)$ is 2-transitive on the points of the unital, there is also contained an element in $\mathrm{PGU}(3,q^2)_{\{P,Q\}}$ which permutes the two points. Thus, $\mid \mathrm{PGU}(3,q^2)_{\{P,Q\}} \mid= 2(q^2-1)$ and $\mid \mathrm{GU}(3,q^2)_{\{P,Q\}} \mid=\mid \mathrm{PGU}(3,q^2)_{\{P,Q\}} \mid\mid Z(\mathrm{GU}(3,q^2)) \mid= 2(q^2-1)(q+1)$. Then, $\widetilde{T} \cdot \widetilde{G}_{\{P,Q\}} \subseteq (T \cdot \widetilde{G})_{\hat{B}}$. We want to prove that $\widetilde{T} \cdot \widetilde{G}_{\{P,Q\}} = (T \cdot \widetilde{G})_{\hat{B}}$. Let $\tau g \in (T \cdot \widetilde{G})_{\hat{B}}$, then $\pi_1^{\tau g} \in \hat{B}$ and $\pi_2^{\tau g} \in \hat{B}$. Suppose that $\pi_1^{\tau g} \neq \pi_1$, then $\pi_1^{\tau g} = \pi_1^{g\tau_1} = \pi_2$ and $\pi_2^{\tau g} = \pi_2^{g\tau_1} = \pi_1$, for a suitable $\tau_1 \in T$. Moreover π_1^g is a plane through O, through S and through ℓ_2, since the translation fix E pointwise, hence $\pi_1^g = \pi_2$. Similarly, $\pi_2^g = \pi_1$. Therefore, $\tau_1 \in \widetilde{T}$ and $g \in \widetilde{G}_{\{\ell_1,\ell_2\}}$. Now, suppose that $\pi_1^{\tau g} = \pi_1$ ad consequently $\pi_2^{\tau g} = \pi_2$. Then, $\pi_1^{\tau g} = \pi_1^{g\tau_1} = \pi_1$. Therefore, $\pi_1^g = \pi_1$ and $\pi_2^g = \pi_2$. Hence, $\tau_1 \in \widetilde{T}$ and $g \in \widetilde{G}_{\{\ell_1,\ell_2\}}$. So, the obtained divisible design has the following parameters,

$$v = q^2(q^3+1), \ s = q^2, \ k = 2 \text{ and}$$

$\lambda_2 = \frac{|T\cdot \mathrm{GU}(3,q^2)|}{|(T\cdot \mathrm{GU}(3,q^2))_{\hat{B}}|}\frac{k(k-1)}{v(v-s)} = \frac{q^6(q^3+1)q^3(q^2-1)(q+1)}{q^2 2(q^2-1)(q+1)}\frac{2}{q^2(q^3+1)q^5} = 1.$

The constructed divisible designs admit, as a point- and block-transitive automorphism group, the group $T\cdot \widetilde{G}$ with $\widetilde{G} \cong \mathrm{GU}(3,q^2)$. (iv) Now, consider the embedded subplane $\mathrm{PG}(2,q)$ of E. The unital $\mathcal{U}$ has $q+1$ points in common with the subplane namely (see [Hr] pg. 148) those satisfying

$$\widetilde{\mathcal{C}} = \begin{cases} X_0^2 + X_1^2 + X_2^2 & = & 0 \\ X_3 & = & 0. \end{cases}$$

This is a conic if p is odd and a secant if p is even. Again, we consider the same R-permutation group as above, and we use Proposition 2.4 to construct a divisible design. Choose the following R-transversal subset of $\Pi(\Omega)$;

$$\overline{B} := \Pi_O(\overline{\Omega}_{\widetilde{\mathcal{C}}}) = \left\{ \ \pi \text{ a plane of } \mathrm{AG}(3,q^2) \ \mid \ \pi\cap E \in \overline{\Omega}_{\widetilde{\mathcal{C}}} \text{ and } O = (0,0,0,1)K \in \pi \right\},$$

where

$$\overline{\Omega}_{\widetilde{\mathcal{C}}} = \left\{ \ell \in \Omega \mid \text{ the tangent point of } \ell \text{ is a point of } \widetilde{\mathcal{C}} \right\}.$$

The case for q even is contained in (ii), so let q be odd. Notice that, $(T\cdot\widetilde{G})_{\overline{B}} = \widetilde{G}_{\overline{B}} = \widetilde{G}_{\overline{\Omega}_{\widetilde{\mathcal{C}}}}$, since $\widetilde{G}$ fixes O, and the tangent lines of $\widetilde{\mathcal{C}}$ do not intersect in a point. Moreover, since every point of $\mathcal{U}$ lies on exactly one tangent line, (see [Hr] p.147), $\widetilde{G}_{\overline{\Omega}_{\widetilde{\mathcal{C}}}} = \widetilde{G}_{\widetilde{\mathcal{C}}}$. Therefore, $\mathrm{PGU}(3,q^2)_{\widetilde{\mathcal{C}}} \cong \mathrm{GU}(3,q^2)_{\widetilde{\mathcal{C}}}/Z(\mathrm{GU}(3,q^2))$, and $\widetilde{G}_{\widetilde{\mathcal{C}}} \cong \mathrm{GU}(3,q^2)_{\widetilde{\mathcal{C}}}$. Thus, we can consider any unital on $\mathrm{PG}(2,q^2)$, with q odd, to find the stabilizer in $\mathrm{PGU}(3,q^2)$ of the points of the unital in $\mathrm{PG}(2,q)$. Take on $\mathrm{PG}(2,q^2)$ the unital $\widetilde{\mathcal{U}}$ with equation $X_0X_2^q + X_0^qX_2 + X_1^{q+1} = 0$. Then, $\widetilde{\mathcal{U}} \cap \mathrm{PG}(2,q)$ is a conic $\hat{\mathcal{C}}$ with equation $X_1^2 + 2X_0X_2 = 0$. It is known, (see [Hu] p. 242), that $\mathrm{PSU}(3,q^2)_{\{(0,0,1)K\}}$ is generated by the following elements;

$$Q(a,b) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & -a^q \\ 0 & 0 & 1 \end{pmatrix} \ \mid \ a,b \in \mathrm{GF}(q^2) \right\} \text{ and}$$

$$H(k) = \left\{ \begin{pmatrix} k^{-q} & 0 & 0 \\ 0 & k^{q-1} & 0 \\ 0 & 0 & k \end{pmatrix} \ \mid \ k \in \mathrm{GF}(q^2)^* \right\}.$$

In case of $\mathrm{GU}(3,q^2)$, one has to consider as well

$$H(k,c) = \left\{ \begin{pmatrix} k^{-q} & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & k \end{pmatrix} \ \mid \ k \in \mathrm{GF}(q^2)^* \text{ and } c^{q+1} = 1 \right\}.$$

We want to find the elements of $\mathrm{GU}(3,q^2)_{\{(0,0,1)K\}}$ fixing the conic $\hat{\mathcal{C}}$. One can show that they are

$$\left\{ Q(a, -\frac{1}{2}a^2) \ \mid \ a \in \mathrm{GF}(q) \right\},$$

and

$$\left\{ H(h,c) \ \mid \ h \in \mathrm{GF}(q^2)^*, \ c = gh \text{ with } g \in \mathrm{GF}(q)^* \text{and } g^2 = (h^{-1})^{q+1} \right\}.$$

Notice that the group generated by these elements is 2-transitive on $\hat{\mathcal{C}} \setminus \{(0,0,1)K\}$. Moreover, the following element,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

is in $\mathrm{GU}(3,q^2)$, fixes $\hat{\mathcal{C}}$, and maps $(0,0,1)K$ onto $(1,0,0)K$. Therefore, $\mathrm{PGU}(3,q^2)_{\hat{\mathcal{C}}}$ is 3-transitive. So, $\mid \mathrm{GU}(3,q^2)_{\widetilde{\mathcal{C}}} \mid = q(q^2-1)(q+1)$. Thus, the constructed divisible design has the following parameters,

$v = q^2(q^3+1)$, $s = q^2$, $k = q+1$, and

$\lambda_2 = \frac{|T \cdot \mathrm{GU}(3,q^2)|}{|(T \cdot \mathrm{GU}(3,q^2))_{\overline{B}}|} \frac{k(k-1)}{v(v-s)} = \frac{q^6(q^3+1)q^3(q^2-1)(q+1)}{q(q^2-1)(q+1)} \frac{q(q+1)}{q^2(q^3+1)q^5} = q^2(q+1).$

The constructed divisible design admits $T \cdot \widetilde{G}$ with $\widetilde{G} \cong \mathrm{GU}(3,q^2)$ as a point- and block-transitive automorphism group.□

References

[BJL] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Bibliographisches Institut, Mannheim - Wien - Zürich, 1985.

[Ce] C. Cerroni, *Divisible designs from semifield planes*, to appear.

[CeSp] C. Cerroni and A.G. Spera, *On divisible designs and twisted field planes*, J. Comb. Designs to appear.

[Ha] H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. 11, (1975), 255 - 369.

[HP] D. R. Hughes and F. C. Piper, *Projective planes*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[Hr] J. W. P. Hirschfeld, *Projective Geometries Over Finite Fields*, Clarendon Press, Oxford, 1979.

[Hu] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[Lü] H. Lüneburg, *Translation planes*, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[Sch] R.-H. Schulz, *Constant weight codes and divisible designs with large automorphism group*, Rendiconti del Cicolo Matematico di Palermo.Serie II, Suppl.53 (1998) 173–188.

[SchSp1] R.-H. Schulz and A. G. Spera, *Divisible designs admitting a Suzuki group as an automorphism group*, Boll. U.M.I. (8) I - B - 3, (1998), 705-714.

[SchSp2] R.-H. Schulz and A. G. Spera, *Construction of divisible designs from translation planes*, Europ. J. Comb. vol. 19, n.4 (1998), 479-486.

[SchSp3] R.-H. Schulz and A. G. Spera, *Automorphisms of constant weight codes and of divisible designs*. Designs, Codes and Cryptography (DESI); to appear.

[Sp1] A. G. Spera, *t - Divisible designs from imprimitive permutation groups*, Europ. J. Comb. 13, (1992), 409 - 417.

[Sp2] A. G. Spera, *On divisible designs and local algebras*, J. Comb. Designs, vol. 3, n.3 (1995), 203 - 212.

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