On Bárány's theorems of Carathéodory and Helly type

Ehrhard Behrends

Abstract. The paper begins with a self-contained and short development of Bárány's theorems of Carathéodory and Helly type in finite dimensional spaces together with some new variants. In the second half the possible generalizations of these results to arbitrary Banach spaces are investigated. The Carathéodory-Bárány theorem has a counterpart in arbitrary dimensions under suitable uniform compactness or uniform boundedness conditions. The proper generalization of the Helly-Bárány theorem reads as follows: if $\mathcal{C}_n, n = 1, \ldots$ are families of closed convex sets in a bounded subset of a separable Banach space X such that there exists a positive ε_0 with $\bigcap_{C\in\mathcal{C}_n}(C)_\varepsilon = \emptyset$ for $\varepsilon < \varepsilon_0$, then there are $C_n \in \mathcal{C}_n$ with $\bigcap_n (C_n)_\varepsilon = \emptyset$ for all $\varepsilon < \varepsilon_0$; here $(C)_\varepsilon$ denotes the collection of all x with distance at most ε to C.

AMS-classification: 46B20, 46B22, 52A20, 52A35;

keywords: Barany, Caratheodory, Helly, Helly-type theorem, Krein-Milman theorem, RNP.

1. Introduction

The simplest version of Bárány's Carathéodory theorem is often illustrated as follows: imagine in the plane three triangles, the first with red, the second with blue and the third with green vertices; if all contain a point z, then it is possible to choose a red, a blue and a green vertex such that z is in the convex hull of these three points. The surprising feature is that even in this innocent two-dimensional setting there seems to be no really simple proof of this combinatorial fact.

The d-dimensional *Carathéodory-Bárány theorem* reads as follows: if Δ_i, $i = 0, \ldots, d$, are finite subsets of $\mathbb{R}^d$ for which the convex hull $\mathrm{co}\,(\Delta_i)$ of Δ_i contains a common point z, then one may choose $x_i \in \Delta_i$ for $i = 0, \ldots, d$ such that z is in $\mathrm{co}\,(\{x_0, \ldots, x_d\})$. By a duality argument one can deduce the following *Helly-Bárány theorem*: if $\mathcal{C}_i, i = 0, \ldots, d$ are finite families of compact convex subsets of $\mathbb{R}^d$ such that $\bigcap_{C\in\mathcal{C}_i} C = \emptyset$ for every i, then it is possible to find $C_i \in \mathcal{C}_i$ with $\bigcap_i C_i = \emptyset$.

These theorems – which obviously contain the classical Carathéodory and Helly theorems as special cases – have been published in 1982 in [3]. Since then a number of refinements and applications have been studied, see the survey article [6], the papers [1], [4], and the references given there. Also, Bárány's theorem has been applied to find selections of multivalued maps in [2], there the investigations from [5] are continued.

The aim of the present paper is two-fold. In the first chapters we want to provide a *short and self-contained proof* of the Bárány theorems together with some variants. The second half is devoted to the study of infinite dimensional generalizations. This just means that we allow arbitrary Banach (or locally convex) spaces and that we pass in the Carathéodory type theorems from the convex to the closed convex hull. The main results are as follows:

- The Carathéodory-Bárány theorem is false in every infinite dimensional space (proposition 4.1)

- The theorem holds in X provided that X' is separable and all $\Delta_n, n = 1, \ldots$ lie in a common bounded set (theorem 4.3).

- It is also true without any restriction on X' if the Δ_n are contained in a fixed compact convex set K; under this condition the theorem even holds for locally convex spaces if the points of K satisfy the first countability axiom with respect to the relative topology (theorem 4.4).

- The Helly-Bárány theorem does not hold in infinite dimensional situations (proposition 5.1).

- If, however, one imposes a uniform boundedness condition, then it has a counterpart in separable spaces (see the abstract; theorem 5.5). For inseparable spaces also this variant fails to hold (proposition 5.6).

The Carathéodory-Bárány theorems are proved by using our finite-dimensional variant of the corresponding theorem. The strategy to show the Helly-Bárány results is – as in the finite-dimensional case – to reduce them by an appropriate duality argument to a Carathéodory-Bárány theorem.

2. Carathéodory-Bárány theorems in $\mathbb{R}^d$

Our way to prove this theorem is essentially the same as that in [3]. In order to keep the paper self-contained we sketch the simple argument.

The starting point is a lemma which states that all rays from a support point of a ball B to the interior of the half space which contains B necessarily meet the interior of B:

Lemma 2.1 *Let $B(z,r)$ be a ball in the finite-dimensional euclidean space and $\hat{x}$ a point on its boundary. Then, for every x such that $\langle z-\hat{x}, x-\hat{x}\rangle > 0$, there is a positive ε such that $||((1-\varepsilon)\hat{x}+\varepsilon x)-z|| < r$.*

Proof. Assume for simplicity that $z=0$ and $r=1$, and consider the function

$$f : \varepsilon \mapsto ||(1-\varepsilon)\hat{x}+\varepsilon x||^2 = \langle (1-\varepsilon)\hat{x}+\varepsilon x, (1-\varepsilon)\hat{x}+\varepsilon x\rangle.$$

We have $f(0)=1$, and the derivative of f at $\varepsilon = 0$ is $2\langle \hat{x}, x-\hat{x}\rangle < 0$. Hence f is smaller than one for small positive ε. □

Now we prove the *cone version of the Carathéodory-Bárány theorem.* We recall that a *cone* in a real vector space X is a convex set which contains with each of its elements all nonnegative multiples. If Δ is a nonvoid subset of X we denote by

$$\mathrm{cone}\,(\Delta) := \{\sum_{i=1}^{n} \lambda_i x_i | n \in \mathbb{N}, x_i \in \Delta, \lambda_i \geq 0\}$$

the cone which is generated by Δ.

Theorem 2.2 *Let Δ_i, $i = 1,\ldots,d$ be subsets of $\mathbb{R}^d$. If a z is contained in all* cone (Δ_i), *then there are $x_i \in \Delta_i$ such that z lies in* cone $(\{x_1,\ldots,x_d\})$.

Proof. We may assume that all Δ_i are finite. Denote, for $x_i \in \Delta_i, i = 1,\ldots,d$, by $r_{x_1,\ldots,x_d}$ the (euclidean) distance between z and the cone generated by $x_1,\ldots,x_d$ (note that these cones, being generated by finite sets, are closed convex sets). The claim is that the minimum r of the $r_{x_1,\ldots,x_d}$ is zero.

We assume that $r > 0$, this will lead to a contradiction as follows. Choose $x_1,\ldots,x_d$ such that $r_{x_1,\ldots,x_d} = r$. The ball $B(z,r)$ meets $K :=$ cone $(\{x_1,\ldots,x_d\})$ at precisely one point $\hat{x}$. The supporting hyperplane H of $B(z,r)$ at $\hat{x}$ intersects K in a facial subcone which is at most $(d-1)$-dimensional, and it follows – by the cone-version of the classical Carathéodory theorem – that $\hat{x}$ lies in a cone which is generated by at most $d-1$ elements from the set $\{x_1,\ldots,x_d\}$.

> The cone version of Carathéodory's theorem states that for $z \in \operatorname{cone}(\Delta) \subset \mathbb{R}^d$ there are $y_1, \ldots, y_d \in \Delta$ with $z \in \operatorname{cone}(\{y_1, \ldots, y_d\})$, it corresponds to the special case $\Delta_1 = \cdots = \Delta_d$ of the theorem. For our proof we need only the $(d-1)$-dimensional case of Carathéodory's theorem, and therefore – since we can argue by induction on d – our approach is in fact self-contained.

Suppose, e.g., that $\hat{x} = \lambda_2 x_2 + \cdots + \lambda_d x_d$. Since z lies in the interior of the set $\{x \mid \langle x_0 - \hat{x}, x - \hat{x}\rangle \geq 0\}$ and also in cone (Δ_1) there must be an $x \in \Delta_1$ with $\langle x_0 - \hat{x}, x - \hat{x}\rangle > 0$. By the preceding lemma it follows that $r_{x,x_2,\ldots,x_d} < r$, a contradiction. □

The following theorem is slightly more general than Bárány's Carathéodory theorem 2.3 in [3]:

Theorem 2.3 *Let Δ_i, $i = 1, \ldots, d$, be subsets of $\mathbb{R}^d$. Also, let x_0 and z be such that z is contained in the convex hull of $\Delta_i \cup \{x_0\}$ for $i = 1, \ldots, d$.*

Then there are $x_i \in \Delta_i$, $i = 1, \ldots, d$, such that z lies in the convex hull of $\{x_0, x_1, \ldots, x_d\}$.

Proof. Suppose without loss of generality that $z = 0$. We assume that z even lies in the *interior* all convex hulls co $(\Delta_i \cup \{x_0\})$ (this can be achieved by adding to the points of Δ_i the unit vectors of $\mathbb{R}^d$, multiplied with a small positive η, and then letting η tend to zero). Therefore we may choose a positive ε such that $-\varepsilon x_0$ also lies in these convex hulls and thus in particular in all cone $(\Delta_i \cup \{x_0\}), i = 1, \ldots, d$. By theorem 2.2, we find $y_i \in \Delta_i \cup \{x_0\}$ and $\lambda_i \geq 0$ with $-\varepsilon x_0 = \lambda_1 y_1 + \cdots + \lambda_d y_d$.

This means that there is a nontrivial linear combination $0 = \mu_0 x_0 + \mu_1 x_1 + \cdots + \mu_d x_d$ with $x_i \in \Delta_i$ for $i = 1, \ldots, d$ and $\mu_i \geq 0$, and one only has to divide by $\mu_0 + \cdots + \mu_d$ to finish the proof. □

Corollary 2.4 *If a point z is contained in the convex hull of each of the subsets Δ_i, $i = 0, \ldots, d$ of $\mathbb{R}^d$, then there are $x_i \in \Delta_i$ such that z lies in the convex hull of $\{x_0, \ldots, x_d\}$.*

Crucial for our further investigations will be the following generalization of the preceding theorem and its corollary (which correspond to the case $C = \{x\}$):

Theorem 2.5 *Let Δ_i, $i = 1, \dots, d$ be subsets of $\mathbb{R}^d$ and $C \subset \mathbb{R}^d$ be convex. Further, let x_0 be a point such that the convex hull of $\Delta_i \cup \{x_0\}$ meets C for all i. Then there are $x_i \in \Delta_i, i = 1, \dots, d$, such that the convex hull of $\{x_0, x_1, \dots, x_d\}$ intersects C.*

In particular it follows that, for subsets Δ_i, $i = 0, \dots, d$ for which each $\mathrm{co}\,(\Delta_i)$ meets C, there are $x_i \in \Delta_i$ with $C \cap \mathrm{co}\,(\{x_0, \dots, x_d\}) \neq \emptyset$.

Proof. Choose $y_i \in C$, $i = 1, \dots, d$, such that $y_i \in \mathrm{co}\,((\Delta_i \cup \{x_0\})$; fix also an arbitrary $y_0 \in C$.

Then 0 lies in the convex hull of $\tilde{\Delta}_i := \{x - y_i | x \in \Delta_i\} \cup \{x_0 - y_i\}$ for $i = 1, \dots, d$. Therefore, by the preceding theorem, 0 can be written as a convex combination $0 = \lambda_0(x_0 - y_0) + \lambda_1 \tilde{x}_0 + \cdots + \lambda_d \tilde{x}_d$ for suitably chosen $\tilde{x}_i \in \tilde{\Delta}_i$. And consequently a convex combination of x_0 and certain $x_i \in \Delta_i, i = 1, \dots, d$ equals a convex combination of the $y_0, \dots, y_d$. □

Some supplements

Is it essential that there are $d+1$ sets Δ_i involved?

The answer is a sound "yes" since even in the original theorem fewer than $d+1$ points in general will not suffice. However, let us reformulate the Carathéodory-Bárány theorem slightly:

> Let x be an element of the convex hull of $\bigcup_{C \in \mathcal{C}_i} C$ for $i = 0, \dots, d$, where each $\mathcal{C}_i$ is a family of convex sets in $\mathbb{R}^d$. Then there are $C_i \in \mathcal{C}_i$ such that $x \in \mathrm{co}\,(\bigcup_{i=0}^d C_i)$.

It might happen that fewer than $d+1$ families $\mathcal{C}_i$ work similarly well:

Proposition 2.6 *Let $\mathcal{C}_i$, $i = 1, \dots, d$ be collections of convex sets in $\mathbb{R}^d$ such that there is an x_0 in $\bigcap_i \bigcap_{C \in \mathcal{C}_i} C$. Then for every z which lies in the convex hull of $\bigcup_{C \in \mathcal{C}_i} C$ for $i = 1, \dots, d$, there are $C_i \in \mathcal{C}_i$ with $z \in \mathrm{co}\,(\bigcup_{i=1}^d C_i)$.*

Proof. The proof follows immediately from theorem 2.3 if one chooses as x_0 in this theorem the x_0 from the common intersection. □

In the case of the classical Carathéodory theorem we can prove a stronger result which seems to have no counterpart for the Bárány generalization:

Proposition 2.7 *Let $C_0, \ldots, C_d$ be compact convex sets in $\mathbb{R}^d$. Then the following assertions are equivalent:*

(i) *For every z in the convex hull D of $\bigcup_{i=0}^d C_i$ there are $x_1, \ldots, x_d$, each x_j lying in some C_i, such that z is a convex combination of the $x_1, \ldots, x_d$.*

(ii) *There is an x_0 in $\bigcap_{i_0=0}^d \operatorname{co}(\bigcup_{i \neq i_0} C_i)$.*

Proof. Suppose that (ii) holds. Then it suffices to apply theorem 2.3 with x_0 from the assumption and $\Delta_1 = \cdots = \Delta_d = \bigcup_i C_i$.

Conversely, suppose that (i) is satisfied. We put $D = \operatorname{co}(\bigcup_i C_i)$ and $D_{i_0} := \operatorname{co}(\bigcup_{i \neq i_0} C_i)$ for $i_0 = 1, \ldots, d$. By assumption, the convex set D is the union of the D_{i_0}, and thus the complement of this union is connected. The complement of the union of fewer than $d+1$ sets D_{i_0} is also connected since every such union is star-shaped. Thus, by Levi's theorem, the D_{i_0} must have a point in common (see [8] or theorem 3.6 in [6]). □

Here is a further result of this kind:

Proposition 2.8 *Let H be a k-dimensional subspace of $\mathbb{R}^d$. Suppose that, for certain finite $\Delta_1, \ldots, \Delta_{d-k}$, a point z is contained in the closed convex hull of $H \cup \Delta_i$ for all i. Then there are $x_i \in \Delta_i$ such that z lies in the closed convex hull of $H \cup \{x_1, \ldots, x_{d-k}\}$.*

Remark: Note that the case $k = 0$ is (essentially) equivalent with theorem 2.3. Note also that "closed convex hull" cannot be replaced by "convex hull" here: simply consider in $\mathbb{R}^2$ a one-dimensional H and a Δ_1 consisting of two points lying in the same halfspace and having the same distance to H.

Proof. Let ω be the canonical quotient map from $\mathbb{R}^d$ to $\mathbb{R}^d/H$. For any finite $\Delta \subset \mathbb{R}^d$, the closed convex hull of $H \cup \Delta$ is just the pre-image under ω of the convex hull of $\omega(\Delta)$. With this observation we may reduce the assertion of the theorem to theorem 2.3 which has to be applied in the $(d-k)$-dimensional space $\mathbb{R}^d/H$ with $x_0 = 0$. □

How many representations as convex combinations are there?

Let $z \in \mathbb{R}^d$ be in the convex hull of r points $x_1, \ldots, x_r$, where $r > d+1$. By Carathéodory's theorem, there are in fact $d+1$ points among the x_i which have z in their convex hull. In how many ways it is possible to choose $d+1$ points with this property?

Assume for definiteness that the x_i are pairwise different. Fix any $d+1$ of them – suppose that these are $x_1, \ldots, x_{d+1}$ – such that z can be written as a convex combination. Now theorem 2.3 comes into play. With $\Delta_1 = \cdots = \Delta_d = \{x_1, \ldots, x_{d+1}\}$ and with $x_0 =$ one fixed x_{i_0} from the $x_{d+2}, \ldots, x_r$ we find a set Δ of $d+1$ elements containing this x_{i_0} and d points from the set $\{x_1, \ldots, x_{d+1}\}$ with z in the convex hull of Δ. *This amounts in $r-d$ different possibilities to represent z.*

In order to obtain a similar result for the number of representations in the Carathéodory-Bárány theorem we suppose that the finite families Δ_i, $i = 0, \ldots, d$ contain n_i elements, respectively, and that z is in the convex hull of each Δ_i. Let n be the maximum of the n_i. Then, by arguing similarly as in the Carathéodory case, we get *n essentially different ways to select families $x_i \in \Delta_i$ having z in their convex hull.*

> Note that we count differently in the Carathéodory and in the Carathéodory-Bárány case: in the former we count subsets whereas in the latter ordered $(d+1)$-tuples are of interest.

3. Helly-Bárány theorems in $\mathbb{R}^d$

We will use the following variant of duality:

Definition 3.1 Let C be a compact convex subset of $\mathbb{R}^d$. By $\widehat{C}$ we denote the set

$$\{(x', a) \mid x' : \mathbb{R}^d \text{ linear}, x'(x) \geq a \text{ for all } x \in C\} \subset (\mathbb{R}^d)' \times \mathbb{R}.$$

It is clear that $\widehat{C}$ is a closed cone with nonempty interior. Crucial for our proof of the Helly-Bárány theorem is the following result. It should be folklore, however, we only have found it in a slightly different form ([10], proposition 5.8).

Proposition 3.2 *Let $C_1, \ldots, C_r$ be compact convex subsets in $\mathbb{R}^d$. Then the following are equivalent:*

(i) $\bigcap_{i=1}^r C_i = \emptyset$;
(ii) $(0,1) \in \widehat{C_1} + \cdots + \widehat{C_r}$.

Proof. Suppose that there exists a z which lies in all C_i. Then, for $(x_i', a_i) \in \widehat{C_i}$, we have $x_i'(z) \geq a_i$ for all i. Thus, if it happens that $x_1' + \cdots + x_r' = 0$, then $0 \geq a_1 + \cdots + a_r$, and therefore it is not possible that $(0,1)$ lies in $\widehat{C_1} + \cdots + \widehat{C_r}$.

Conversely, suppose that the intersection of the C_i is empty. This just means that $C := C_1 \times \cdots \times C_r$ does not meet the subspace $Y := \{(x,\ldots,x) | x \in \mathbb{R}^d\}$ of $(\mathbb{R}^d)^r$. Since C is compact and convex, the Hahn-Banach separation theorem provides a functional Φ which is strictly positive on C and which vanishes on Y. Φ has the form $(x_1,\ldots,x_r) \mapsto x_1'(x_1) + \cdots + x_r'(x_r)$, and our assumptions imply that $x_1' + \cdots + x_r' = 0$ and also that $x_1'(x_1) + \cdots + x_r'(x_r) \geq c$ for a strictly positive c and all $x_1 \in C_1, \ldots, x_r \in C_r$. Thus, with $a_i := \min x_i'(C_i)$, it follows that $c' := a_1 + \cdots + a_r \geq c$ and $(x_i, a_i) \in \widehat{C_i}$. Consequently, $(0, c')$ and thus also $(0,1)$ lie in the cone $\widehat{C_1} + \cdots + \widehat{C_r}$. □

We now prove the Helly-Bárány theorem:

Theorem 3.3 *Let $\mathcal{C}_i$ be families of compact convex sets in $\mathbb{R}^d$ for $i = 0, \ldots, d$ such that $\bigcap_{C \in \mathcal{C}_i} C = \emptyset$ for all i. Then there are $C_i \in \mathcal{C}_i$ such that*

$$\bigcap_{i=0}^{d} C_i = \emptyset.$$

Proof. By the compactness of the sets under consideration we may assume that all $\mathcal{C}_i$ are finite. From proposition 3.2 it follows that we may choose $y_C^i \in \widehat{C}$ for the $C \in \mathcal{C}_i$ such that $(0,1) = \sum_{C \in \mathcal{C}_i} y_C^i$. We define Δ_i to be the collection of the y_C^i, $C \in \mathcal{C}_i$. By theorem 2.2 we find $y_{C_i}^i \in \Delta_i$ such that $(0,1)$ lies in the cone which is generated by these elements. A fortiori, $(0,1)$ lies in $\widehat{C_1} + \cdots + \widehat{C_r}$, and, with another appeal to proposition 3.2, the proof is complete. □

As a *supplement* we ask:
In how many ways it is possible to produce an empty intersection?

We want to stress the difference to the Carathéodory-Bárány case. There one could work with theorem 2.3, by which we had at our disposal the "free" point x_0. In the present case we have to deal with theorem 2.2, and thus the situation is different.
In fact, the following *example* shows that it might happen that *only one choice is suitable.*

Define in $\mathbb{R}^2$ three collections of compact convex sets as follows:

- $\mathcal{C}_1$ consists of the two rectangles spanned by $(0.25, -1), (0.25, 1), (1, 1), (1, -1)$ resp. by $(-1, -1), (-1, +1), (-0.25, 1), (-0.25, -1)$.

- The C in $\mathcal{C}_2$ are the rectangles generated by $(-1,1),(1,1),(-1,0.25),(1,0.25)$ resp. by $(1,-1),(-1,-1),(1,-0.25),(-1,-0.25)$.
- Finally, $\mathcal{C}_3$ consists of the one-point set $\{(1,1)\}$ and the square with the vertices $(\pm 0.5, \pm 0.5)$.

Then the conditions of the theorem are met, and there is precisely one choice to have an empty intersection. (Similar examples can be constructed in every dimension.)

4. Carathéodory-Bárány theorems in arbitrary Banach spaces

We now turn from $\mathbb{R}^d$ to an the infinite-dimensional situation, our sets will be subsets of an arbitrary Banach space X or a locally convex space E.

Clearly a Carathéodory theorem is *not* to be expected, finite convex combinations with a fixed number of summands will never suffice in the infinite dimensional setting. Therefore we pass from elements of the convex hull to points in the *closed convex hull.* A naive translation of the Carathéodory theorem then has to start with a subset Δ of X (or E) and a point z in the closed convex hull $\overline{\mathrm{co}}\,(\Delta)$ of Δ, and the question is: how many points $x_i, i \in I$ have to chosen from Δ such that z lies in the closed convex hull of $\{x_i \mid i \in I\}$? The answer is trivial: countably many x_i will do in the Banach space case, and for general E the cardinality of I has to be as big as the smallest cardinality of a base of the neighbourhoods of zero.

Thus the Carathéodory theorem does not lead to interesting generalizations, but this is not obvious for the Carathéodory-Bárány theorem. In the case of Banach spaces the following *natural problem* arises:

> Let $\Delta_n, n = 1, \ldots,$ be subsets of a Banach space X such that a certain point z lies in the closed convex hull of Δ_n for every n. Is it possible to choose $x_n \in \Delta_n$ such that z lies in $\overline{\mathrm{co}}\,(\{x_1, x_2, \ldots\})$?

Our main results are that the answer is "no" in general and that it is in the positive if the Δ_n are uniformly "not too big".

Proposition 4.1 *Let X be an infinite dimensional real Banach space. Then there are subsets Δ_n, $n = 1, \ldots$ such that:*

(i) 0 *lies in the convex hull of each* Δ_n*, but*

(ii) *it is not possible to choose $x_n \in \Delta_n$ in such a way that* 0 *lies in the closed convex hull of* $\{x_1, x_2, \ldots\}$.

Remark: A similar counterexample to the "unrestricted" Carathéodory-Bárány theorem can be constructed in arbitrary infinite-dimensional locally convex Hausdorff spaces.

Proof. Denote by B be the closed unit ball of X. Our construction will be based on the following

Claim: Let Γ_i, $i = 1, \ldots, r$, be finite subsets of X such that for every i the convex hull of Γ_i does not intersect B. Then there exists an $x \in X$ such that the convex hulls of the $2r$ sets $\Gamma_i \cup \{x\}$, $\Gamma_i \cup \{-x\}$ have an empty intersection with B.

Proof of the claim: Choose, for $i = 1, \ldots, r$, a continuous functional x_i' such that $||x_i'|| = 1$ and $x_i'(x) \geq c_i$ for all $x \in \Gamma_i$ and some $c_i > 1$. Let z be any nonzero element of X which lies in the kernel of all x_i', we claim that $x = az$ for "sufficiently large" a has the desired properties.

To this end, choose a number b which strictly dominates all $1/c_i$ and which is strictly smaller than 1. Let $a \in \mathbb{R}$ be arbitrary, y an element of the convex hull of Γ_i for some i, $\lambda \in [0, 1]$, and $w := \lambda y \pm (1 - \lambda) az$ a point in the convex hull of $\Gamma_i \cup \{az\}$ or $\Gamma_i \cup \{-az\}$. If $\lambda \geq b$, then $x_i'(w) = x_i'(\lambda y) \geq bc_i > 1$ so that w does not lie in B. For the λ with $\lambda \leq b$, however, we know that the norm of w can be estimated from below by $|a|(1 - b)||z|| - b||y||$. This number will be greater than one uniformly in y for large a since the convex hulls of the Γ_i are bounded, and this proves the claim.

We now turn to the proof of the propostion. Start with any x_1 not lying in B and put $\Delta_1 := \{-x_1, x_1\}$. By the claim, with $\Gamma_1 := \{x_1\}, \Gamma_2 := \{-x_1\}$, we get an x_2 such that the convex hulls of $\{\pm x_1, \pm x_2\}$ do not meet B; the second set, Δ_2, consists of $-x_2$ and x_2. The four sets $\operatorname{co}(\{\pm x_1, \pm x_2\})$ are the new Γ's for an another application of the claim: we get x_3 such that $\operatorname{co}(\{\pm x_1, \pm x_2, \pm x_3\}) \cap B = \emptyset$, and Δ_3 is the set $\{-x_3, x_3\}$. It should be clear how to proceed, and the Δ_i which we have constructed in this way clearly have the claimed properties. $\square$

To get positive results we will have to impose certain restrictions on the Δ_n. A natural approach is to suppose that they are *uniformly bounded.* This, however, will not suffice in general:

Proposition 4.2 *There exists a Banach space X which contains subsets Δ_n, $n = 1, \ldots$ in the unit ball such that*

(i) 0 *lies in the convex hull of each* Δ_n.
(ii) *for* no *choice of* $x_n \in \Delta_n$ *one has* $0 \in \overline{\text{co}}\,(\{x_1, x_2, \ldots\})$.

Proof. This is simple: one only has to consider $X = l^1$, the space of absolutely summable sequences, together with $\Delta_n := \{-e_n, e_n\}$, where e_n denotes the canonical n'th unit vector. □

Therefore a "bounded" Carathéodory-Bárány theorem fails in every space which contains l^1. The next theorem states that counterexamples do not exist if the dual space X' of X is not too big:

Theorem 4.3 *Let X be a Banach space such that X' is separable (in fact it will suffice that every separable subspace of X has a separable dual, that is that X' has RNP). Then, if Δ_n, $n = 1, \ldots$ are uniformly bounded subsets such that a certain point z lies in the closed convex hull of every Δ_n, then there are $x_n \in \Delta_n$ with $z \in \overline{\text{co}}\,(\{x_1, x_2, \ldots\})$.*

Proof. Without loss of generality we may suppose that $z = 0$ and that all Δ_n lie in the unit ball of X and are at most countable. Thus they are contained in a separable subspace Y, and therefore it is no restriction to replace the condition "X' is separable" with "X' has RNP".

The crucial idea is to apply the finite-dimensional Carathéodory-Bárány theorem by using functionals. Let $x'_1, \ldots, x'_r$ be arbitrary in the dual unit ball, and $\varepsilon > 0$. We consider the map

$$\Phi : X \to \mathbb{R}^r, x \mapsto (x'_1(x), \ldots, x'_r(x)),$$

and we pass from the Δ_n to the $\tilde{\Delta}_n := \Phi(\Delta_n)$. The assumption implies that $0 \in \mathbb{R}^r$ lies in the closed convex hull of $\tilde{\Delta}_n$ for all n. Thus, if we define C to be the ball in $\mathbb{R}^r$ of radius ε with respect to the maximum norm, we know that the convex hull of $\tilde{\Delta}_n$ meets C for every n. In particular this holds for the $r+1$ convex hulls of $\tilde{\Delta}_n$, $n = 1, \ldots, r+1$, and we may choose – by theorem 2.5 – points $x_i \in \Delta_i$ such that the convex hull K of $\{\Phi(x_1), \ldots, \Phi(x_{r+1})\}$ intersects C. And since we are dealing with the maximum norm this means that the minimum of x'_i on K is a most ε.

This construction will now be applied as follows. Choose a dense sequence $x'_1, x'_2, \ldots$ in the dual unit ball of X, and then:

- Choose $x_1 \in \Delta_1, x_2 \in \Delta_2$ such that the minimum of x'_1 on the convex hull of x_1, x_2 is $\leq 1/2$.

- Then select $x_n \in \Delta_n$, $n = 3, 4, 5$ such that both x'_1 and x'_2 assume values $\leq 1/3$ on the convex hull of $\{x_3, x_4, x_5\}$.
- In the next step the above construction provides $x_n \in \Delta_n$, $n = 6, 7, 8, 9$ such that the three functionals x'_1, x'_2, x'_3 have a minimum which is $\leq 1/4$ on the convex hull of these four elements.

It should be clear that we get in this way a sequence (x_n) with $x_n \in \Delta_n$ such that the infimum of all x'_i is ≤ 0 on the convex hull. It follows then, by the uniform boundedness of the Δ_n and the denseness of the $x'_1, x'_2, \ldots$, that there doesn't exists any x' which strictly separates 0 from the closed convex hull of $\{x_1, x_2, \ldots\}$, and this means by the Hahn-Banach theorem that 0 is in fact contained in this closed convex hull. □

With the same technique one can show the following more general assertion:

Theorem 4.4 *Let E be a locally convex Hausdorff space together with a family $(x'_i)_{i \in I}$ of continuous linear functionals which is dense in X' with respect to uniform convergence on bounded sets. If Δ_i are subsets of E for every $i \in I$ which all are contained in the same bounded set B and which all have a point z in their closed convex span, then there are $x_i \in \Delta_i$ such that $z \in \overline{\mathrm{co}}\,(\{x_i \mid i \in I\})$.*

If we pass to the more restrictive assumption that the Δ_n are uniformly compact then the Carathéodory-Bárány theorem holds in every Banach space. In view of the applications we have in mind we consider a slightly more general locally convex situation. Those who are only interested in Banach spaces might replace E by a Banach space X, the second of the two conditions of the theorem then will automatically be satisfied.

Theorem 4.5 *Let E be a locally convex Hausdorff space, $K \subset E$ convex and compact and $\Delta_n \subset K$ for $n = 1, \ldots$. We suppose that z is a point of K such that*

(i) *z lies in the closed convex hull of each Δ_n, and*

(ii) *z has a countable basis of neighbourhoods in the relative topology of K.*

Then there are $x_n \in \Delta_n$ such that z is in the closed convex hull of $\{x_1, x_2, \ldots\}$.

Proof. To prepare our construction we start with any convex neighbourhood U of z. Since K is compact, there are $y_1, \dots, y_r$ in K such that the sets $y_i + U$ cover K. Choose subsets $\tilde{\Delta}_n \subset \Delta_n$ such that

$$\tilde{\Delta}_n + U \subset \Delta_n, \; \Delta_n \subset \tilde{\Delta}_n + U$$

for $n = 1, \dots, r+2$. By assumption, $z + U$ meets the convex hull of Δ_n, and therefore $z + 2U$ will intersect the convex hull of $\tilde{\Delta}_n$. Let Y be the (at most) $(r+1)$-dimensional subspace of X which is spanned by z and the y_i. We apply theorem 2.5 with $C = (z + 2U) \cap Y$. This theorem provides $z_n \in \tilde{\Delta}_n$ for $n = 1, \dots, r+2$ such that their convex hull meets C. Thus, if we choose $x_n \in \Delta_n$ with $z_n \in x_n + U$, it follows that $(z + 3U) \cap \mathrm{co}\,(\{x_1, \dots, x_{r+2}\}) \neq \emptyset$.

Our assumption implies that there are convex neighbourhoods of zero, $U_1, U_2, \dots$ such that z is in a closure of a subset Δ of K provided that $z + 3U_k$ meets Δ for every k. To begin with, choose suitable $x_n \in \Delta_n$ for $n = 1, \dots, r_1$ according to the above construction such that $(z + 3U_1) \cap \mathrm{co}\,(\{x_1, \dots, x_{r_1}\}) \neq \emptyset$, at next $x_n \in \Delta_n$ for $n = r_1 + 1, \dots, r_2$ such that $(z + 3U_2) \cap \mathrm{co}\,(\{x_{r_1+1}, \dots, x_{r_2}\}) \neq \emptyset$, and so on. Then the convex hull of $\{x_1, x_2, \dots\}$ will have z in its closure, and the proof is complete. □

Remarks: 1. If the second condition of the preceding theorem is not met one could nevertheless use the same technique to prove the following weaker statement: if $(U_i)_{i \in I}$ is a neighbourhood base of z in K and $z \in \overline{\mathrm{co}}\,(\Delta_i)$ for certain subsets Δ_i of the compact convex set K for all $i \in I$, then $x_i \in \Delta_i$ exist with $z \in \overline{\mathrm{co}}\,(\{i \mid i \in I\})$.

2. Both theorem 4.3 and theorem 4.5 allow formally more general versions of the form: if the closed convex hulls of the Δ_n meet a fixed closed convex set C, then $C \cap \overline{\mathrm{co}}\,(\{x_1, \dots, \})$ for suitable $x_n \in \Delta_n$.

3. A combination of the Krein-Milman theorem with our Carathéodory-Bárány theorem 4.5 leads to the following assertion: if compact convex sets $K_1, \dots$ of a Banach space are contained in a fixed compact and convex set K, then there are, for every $z \in \bigcap_{n=1}^{\infty} K_n$, extreme points x_n of K_n for $n = 1, \dots$ such that z is in the closed convex hull of $\{x_1, x_2, \dots\}$.

4. We have shown that the Carathéodory-Bárány theorem fails to hold in spaces which contain l^1, and theorem 4.3 states that it is true for spaces with a separable dual. However, there are only rather complicated examples X which don't contain l^1 and nevertheless have an inseparable dual (see [7]), and thus theorem 4.3 is essentially sharp.

5. Helly-Bárány theorems in arbitrary Banach spaces

Let X an arbitrary Banach space and $\mathcal{C}_n$, $n = 1, \ldots$ families of bounded, closed and convex subsets of X; suppose that all intersections $\bigcap_{C \in \mathcal{C}_n} C$ are empty. Then one might ask whether a Helly-Bárány theorem similar to theorem 3.3 holds: do there exist $C_n \in \mathcal{C}_n$ such that $\bigcap_{n=1}^{\infty} C_n = \emptyset$?

In general the answer is "no", even for compact convex sets:

Proposition 5.1 *In every infinite dimensional Banach space there exist families $\mathcal{C}_n$ of compact convex sets with $\bigcap_{C \in \mathcal{C}_n} C = \emptyset$ such that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ for each choice $C_n \in \Delta_n$ for $n = 1, \ldots$. The sets under consideration can even be chosen as subsets of a fixed compact convex set.*

Proof. At first we consider the Banach space l^1. We put

$$K := \{(t_k) \mid |t_k| \leq 1/n^2 \text{ for } k = 1, \ldots\},$$

and for every n we consider the family $\mathcal{C}_n$ consisting of the two sets

$$C_n^{\pm} := \{(t_k) \in K \mid t_n = \pm 1/n^2\}.$$

These $\mathcal{C}_n$ obviously have the claimed properties.

The case of general X can be treated by choosing a basic sequence (x_n) of normalized vectors in X and by mapping l^1 into X via the mapping $\Phi : (t_k) \mapsto \sum t_k x_k$; then the Φ-images of the preceding compact convex sets provide the desired example. $\square$

Thus, in order to be able to prove positive results, we will need certain restrictions. Our approach will use the condition that the empty intersections $\bigcap_{C \in \mathcal{C}_n} C$ are in a sense "uniformly empty". This can be made precise with the help of the following

Definition 5.2 Let $\mathcal{C}$ be a family of closed, bounded and convex sets of a Banach space X such that $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Then, if $\varepsilon > 0$, we write

$$\bigcap_{C \in \mathcal{C}} C =_{\varepsilon} \emptyset$$

provided that $\bigcap_{C \in \mathcal{C}} (C)_{\varepsilon} = \emptyset$;
here $(C)_{\varepsilon}$ denotes the set $(C)_{\varepsilon} := \{x \mid d(x, C) \leq \varepsilon\}$.

If the members of $\mathcal{C}$ are even compact, then the $(C)_\varepsilon$ are the sets $C+B_\varepsilon$ (with B_ε = the ball with radius ε). In this case the preceding ε-variant of empty intersection is automatically satisfied if the $C \in \mathcal{C}$ have no point in common. This simple observation is included to stress that theorem 5.5. below is a generalization of the finite-dimensional case:

Lemma 5.3 *If $\bigcap_{C\in\mathcal{C}} C = \emptyset$, then there is an $\varepsilon > 0$ such that $\bigcap_{C\in\mathcal{C}} C =_\varepsilon \emptyset$.*

Proof. We may assume that $\mathcal{C}$ is finite, $\mathcal{C} = \{C_1, \ldots, C_r\}$. The assertion now follows from the fact that the compact set $C_1 \times \cdots \times C_r \subset X^r$ does not meet the subspace $\{(x, \ldots, x) \mid x \in X\}$ and thus has a strictly positive distance. $\square$

Our main result will be proved by using a duality argument. This will be prepared in the folllowing proposition which can be thought of as a quantitative version of proposition 3.2:

Proposition 5.4 *Let C be nonvoid, closed and convex in the unit ball B of a real Banach space X. By $\tilde{C}$ we denote the following subset of $X' \times \mathbb{R}$ (in this product X' will be provided with the weak-*-topology):*

$$\{(x', a) \mid ||x'|| \leq 1, x'(x) \geq a \text{ for every } x \in C, a \geq -1\}.$$

Then, for a family $\mathcal{C}$ of nonvoid, closed and convex subsets of B and $\varepsilon_0 > 0$, the following assertions are equivalent:

(i) $\bigcap_{C\in\mathcal{C}} C =_\varepsilon \emptyset$ *for every* $\varepsilon < \varepsilon_0$;

(ii) $(0, \varepsilon_0)$ *lies in the closed convex hull of* $\bigcup_{C\in\mathcal{C}} \tilde{C}$.

Proof. Suppose first that, for some $\varepsilon < \varepsilon_0$, there is an x which lies in every $(C)_\varepsilon$. It has to be shown that $(0, \varepsilon_0)$ has a neighbourhood which does not meet the convex hull of the $\tilde{C}$. We claim that

$$U := \{(x', a) \mid x'(x) < \tau, a > \varepsilon_0 - \tau\}$$

has the desired properties, where $\tau := (\varepsilon_0 - \varepsilon)/4$.

In fact, let $C_1, \ldots, C_r \in \mathcal{C}$ be arbitrary and $(x', a) = \sum_{i=1}^r \lambda_i (x'_i, a_i)$ an element of the convex hull of $\bigcup_{i=1}^r \tilde{C}_i$. Suppose that (x', a) lies in U. We choose $x_i \in C_i$ and b_i with $||b_i|| \leq \varepsilon + \tau$ and $x = x_i + b_i$, and we derive a contradiction as follows:

$$\begin{aligned} \tau &\geq \sum_{i=1}^{r} \lambda_i x_i'(x) \\ &= \sum_{i=1}^{r} \lambda_i x_i'(x_i + b_i) \\ &\geq \sum_{i=1}^{r} \lambda_i (a_i - \varepsilon - \tau) \\ &\geq \varepsilon_0 - 2\tau - \varepsilon. \end{aligned}$$

Conversely, let's assume that $(0, \varepsilon_0)$ fails to be in the closed convex hull K of the union of the $\tilde{C}$. Choose an $\varepsilon < \varepsilon_0$ such that $(0, \varepsilon)$ is not contained in K and separate K strictly from this point by a continuous linear funcional Φ. Since X' is provided with its weak-*-topology, Φ has the form $(x', a) \mapsto x'(x) + \eta a$ for suitable $x \in X$, $\eta \in \mathbb{R}$. Strict separation means that $\eta\varepsilon > x'(x) + \eta a$ for all $(x', a) \in \tilde{C}$ for arbitrary $C \in \mathcal{C}$. Since $(0,0)$ lies in K, we know that $\eta > 0$, and we may and will assume that $\eta = 1$.

We claim that $-x$ lies in all $(C)_\varepsilon$. Suppose that this were not the case. This time we separate the closed convex set $(C)_\varepsilon$ by a norm one functional x' from $-x$, that is we choose this x' such that $x'(-x) < c \leq x'((C)_\varepsilon)$. Then, with $a := \inf x'(C)$, we have $x'(-x) < c \leq a - \varepsilon$ or $\varepsilon < x'(x) + a$. This clearly contradicts the above choice of x since $(x', a) \in \tilde{C}$. □

With these preparations at hand we can prove the main result of this chapter:

Theorem 5.5 *Let X be a separable Banach space and $\mathcal{C}_n$ a family of non-void, closed and convex subsets of the unit ball B for every n. Suppose that there is a positive ε_0 such that $\bigcap_{C \in \mathcal{C}_n} C =_\varepsilon \emptyset$ for every n and every $\varepsilon < \varepsilon_0$. Then there are $C_n \in \mathcal{C}_n$, $n = 1, \ldots$, such that $\bigcap_{n=1}^{\infty} C_n =_\varepsilon \emptyset$ for all $\varepsilon < \varepsilon_0$.*

Proof. Since X is separable, every point of the the dual unit ball has a countable basis of neighbourhoods with respect to the relative weak-*-topology. By this remark we may apply theorem 4.5, the compact set K from this theorem is the product of the dual unit ball (which is compact by the Alaoglu-Bourbaki theorem) with $[-1, 1]$, z is the point $(0, \varepsilon_0)$, and the Δ_n are the $\bigcup_{C \in \mathcal{C}_n} \tilde{C}$. In this way the preceding theorem is reduced to theorem 4.5 and proposition 5.4 (similarly to the finite-dimensional case, where theorem 3.3 was a consequence of proposition 3.3 and theorem 2.2). □

Remarks: 1. If all $\mathcal{C}_n$ in the theorem contain only two sets, C_n^0 and C_n^1, say, then one can prove the theorem without using proposition 5.4.

Assume that, for some $\varepsilon < \varepsilon_0$, the intersections $\bigcap_{n=1}^{\infty} C_n^{\delta_n}$ are empty for arbitrary sequences (δ_n) in $\{0,1\}$. Let us choose $x_{\delta_1,\delta_2,\ldots}$ from each such intersection. Since, by assumption, $(C_n^0)_{\varepsilon+\tau} \cap (C_n^1)_{\varepsilon+\tau} = \emptyset$ for $\tau := (\varepsilon_0 - \varepsilon)/2$ it follows that each two different $x_{\delta_1,\delta_2,\ldots}$ have a distance of at least 2τ. Now there are uncountably many of them, and this contradicts the separability of X.

In fact this argument shows that all but at most countably many of the $\bigcap_{n=1}^{\infty} C_n^{\delta_n}$ are nonempty. It would be interesting to have a similar cardinality argument for the case of arbitrary $\mathcal{C}_n$.
2. In particular it follows from $\bigcap_{C \in \mathcal{C}_n} C =_{\varepsilon_0} \emptyset$ for all n that one finds, for $\varepsilon < \varepsilon_0$, sets $C_n \in \mathcal{C}_n$ with $\bigcap_{n=1}^{\infty} C_n =_{\varepsilon} \emptyset$. It is not clear, however, whether this holds also for $\varepsilon = \varepsilon_0$. The main problem is that – in contrast to the finite dimensional situation – it does not follow from $\bigcap_{C \in \mathcal{C}} C =_{\varepsilon_0} \emptyset$ that $\bigcap_{C \in \mathcal{C}} C =_{\varepsilon_0+\tau} \emptyset$ for some positive τ; cf. the example on page 61 in [9].

It remains to investigate the role of the separabilty of X, is this property essential or is it only imposed to make an application of theorem 4.5 possible? Here is an example of an inseparable space where no Helly-Bárány result like that of theorem 5.5 can be proved:

Proposition 5.6 *In l^∞, the space of bounded sequences, there are families $\mathcal{C}_n$ of closed convex subsets of the unit ball such that $\bigcap_{C \in \mathcal{C}_n} C =_{\varepsilon} \emptyset$ for a suitable positive ε and all n, but $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ for every choice $C_n \in \mathcal{C}_n$.*

Proof. The counterexample is similar to that from the beginning of this chapter: define $\mathcal{C}_n$ to be the collection of the two sets $C_n^{\pm}$, where

$$C_n^{\pm} := \{(x_k) \in l^\infty \mid ||(x_k)|| = 1, x_n = \pm 1\}.$$

□

Acknowledgement: the author wishes to express his gratitude to K. Nikodem who provided the manuscript [2], it gave rise to the present investigations.

References

[1] N. Alon, G. Kalai. *Bounding the piercing number.* Discrete and computational geometry **13** (1995), 245–256.

[2] M. Balaj, K. Nikodem. *Remarks on Bárány's theorem and affine selections.* (Preprint)

[3] I. BÁRÁNY. *A generalization of Carathéodory's theorem.* Discrete Mathematics **40** (1982), 141–152.

[4] I. BÁRÁNY. *Carathéodory's theorem, colourful and applicable.* Bolya Soc., Math. Studies **6** (1997), 11-21.

[5] E. BEHRENDS, K. NIKODEM. *A selection theorem of Helly type and its applications.* Studia Math. **116** (1995), 43–48.

[6] J. ECKHOFF. *Helly, Radon, and Carathéodory type theorems.* in: Handbook of convex geometry, edited by P.M. Gruber and J.M. Wills (1993), Elsevier Science Publishers, 389–448.

[7] R. C. JAMES. *A separable somewhat reflexive Banach space with non-separable dual.* Bull. of the AMS **80** (1974), 738–743.

[8] F.W. LEVI. *Eine Ergänzung zum Hellyschen Satze.* Arch. Math. **4** (1953), 222–224.

[9] J. LINDENSTRAUSS. *Extension of compact operators.* Mem. of the AMS **48** (1964).

[10] F. A. VALENTINE. *Convex sets.* McGraw Hill, reprinted by R.E. Krieger (1976).

I. Mathematisches Institut, Freie Universität Berlin, Arnimallee 2–6,
D-14 195 Berlin, Germany; e-mail: behrends@math.fu-berlin.de