MULTIGRID METHODS BASED ON THE UNCONSTRAINED PRODUCT SPACE ARISING FROM MORTAR FINITE ELEMENT DISCRETIZATIONS

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Abstract. The mortar finite element method allows the coupling of different discretizations across subregion boundaries. In the original mortar approach, the Lagrange multiplier space enforcing a weak continuity condition at the interfaces is defined as a modified finite element trace space. Here, we present a new approach, where the Lagrange multiplier space is replaced by a dual space without loosing the optimality of the a priori bounds. Using the biorthogonality between the nodal basis functions of this Lagrange multiplier space and a finite element trace space, we derive an equivalent symmetric positive definite variational problem defined on the unconstrained product space. The introduction of this formulation is based on a local elimination process for the Lagrange multiplier. This equivalent approach is the starting point for the efficient iterative solution by a multigrid method. To obtain level independent convergence rates for the $\mathcal{W}$–cycle, we have to define suitable level dependent bilinear forms.

Key words. mortar finite elements, Lagrange multiplier, dual space, non–matching triangulations, multigrid methods, level dependent bilinear forms

AMS subject classifications. 65N15, 65N30, 65N55

1. Introduction. The framework of mortar methods [Ben95, BMP93, BMP94] provides a powerful tool to analyze domain decomposition techniques based on the coupling of different discretization schemes or of nonmatching triangulations along interior interfaces. Flexible discretization schemes can be derived for time dependent problems, diffusion coefficients with jumps, problems with local anisotropies as well as corner singularities without loosing the optimality of the a priori bounds. The characteristic idea is to replace the pointwise continuity across the interfaces by a weaker one. Recently, a lot of work has been done on efficient iterative solvers for the arising linear systems [AK95, AKP95, AMW96, AMW99, BD98, BDH99, BDL99, BDW98, CW96, Dry98a, Dry98b, EHI+97, EHI+98, GP98, HIK+98, Kuz95a, Kuz95b, Kuz98, KW95, Lac98, WW98, WW99]. All these techniques are either based on the positive definite variational problem on the constrained nonconforming space or on the equivalent saddle point problem [Ben95]. In the first case very often, the mortar projection has to be applied within each smoothing step and requires the solution of a mass–matrix system. In case that the saddle point problem forms the starting point for the construction of the solver, a Schur complement system plays an essential role. There are different approaches. One possibility is to use a good preconditioner for the exact Schur complement as introduced in [Kuz95a, Kuz95b]. Following an idea of Braess and Sarazin given for the Stokes problem [BS97], a modified Schur complement system has to be solved exactly within each smoothing step [BD98, BDW98, WW98]. More recently, this idea is generalized for the mortar setting in [WW99] using ideas introduced in [Zul98]. The exact solution of the modified Schur complement system is replaced by an iterative one resulting in an inner and outer iteration cycle. Here, we introduce a new equivalent mortar setting. We combine the idea of dual basis functions with standard multigrid techniques for symmetric positive definite systems.

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Replacing the modified trace space in the definition of the Lagrange multiplier by a locally defined dual one yields a priori estimates of the same order [Woh98]. As a consequence of the biorthogonality between the nodal basis functions of the trace space and the Lagrange multiplier space, the mass–matrix on the non–mortar side is reduced to a diagonal one. Based on these observations, we introduce a new mortar formulation associated with the unconstrained product space. Starting with the saddle point problem, the Lagrange multiplier can be locally eliminated, and we finally obtain a positive definite problem on the product space. Both, the algebraic and the variational, formulations are discussed and analyzed. The new setting is the starting point for the introduction of our multigrid method for mortar finite elements. In particular, we have not to solve exactly or iteratively a modified Schur complement system and not to solve a tridiagonal mass matrix system in each smoothing step.

The rest of this paper is organized as follows: In Section 2, we give the abstract mortar formulation as positive definite variational problem on the constrained space as well as the equivalent saddle point formulation. Here, we consider different choices of the Lagrange multiplier spaces providing optimal a priori error estimates. In particular, we focus on the introduction of a dual basis including an example for the quadratic case in 2D and the linear case in 3D. Section 3 is devoted to the introduction of the new equivalent mortar formulation based on the unconstrained product space. Here, we will proof equivalence and positive definiteness of the new formulation. In Section 4, we present our multigrid method based on standard techniques. To obtain a suitable approximation property, the natural restriction operator has to be replaced by a level dependent one. A special type of smoother including a standard Gauß–Seidel one will be considered. Level independent convergence rates are obtained for the $\mathcal{W}$–cycle provided that the number of smoothing steps is large enough. Finally in Section 5, we present numerical results for the $\mathcal{V}$– and $\mathcal{W}$–cycle. The performance of the multigrid cycle is shown for several test examples including discontinuous coefficients and a corner singularity.

2. A modified mortar method using dual spaces. In this section, the nonconforming positive definite and the equivalent saddle point formulation of the mortar method are given. Suitable Lagrange multiplier spaces yielding optimal a priori estimates are presented. We focus on the construction of dual basis functions guaranteeing a local support of the nodal basis functions of the constrained space. Dual basis functions for mortar finite elements are first used in [Woh98], where only the first order case is considered. This special choice of Lagrange multiplier space gives rise to a diagonal mass–matrix associated with the non–mortar side. In addition to the dual basis introduced in [Woh98], we consider quadratic Lagrangian finite elements in 2D and the first order case in 3D. Optimal a priori estimates for the weak solution in both the L^2– and H^1–norm as well as for the Lagrange multiplier in the $H_{00}^{1/2}$–dual norm and a weighted L^2–norm are obtained. For the standard approach, we refer to [Ben95, BM95, BMP93, BMP94]. The introduction of the mesh dependent norm for the Lagrange multiplier can be found in [Woh97] and the a priori estimates in case of a dual basis in [Woh98].

Let us first give the problem setting. We consider the model problem

$$\begin{aligned} Lu := -\operatorname{div}(a\nabla u) + b\,u &= f \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \Gamma := \partial\Omega\,, \end{aligned} \tag{2.1}$$

where Ω is a bounded, polygonal domain in $\mathbb{R}^2$and $f \in L^2(\Omega)$. Let $a \in L^\infty(\Omega)$ be an uniformly positive function and let $0 \le b \in L^\infty(\Omega)$. We assume the domain Ω has been decomposed into K non–overlapping polyhedral subdomains Ω_k by

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \quad \text{and } \Omega_l \cap \Omega_k = \emptyset, \ \ k \neq l\,.$$

Each subdomain Ω_k is associated with a family of shape regular simplicial triangulations $\mathcal{T}_{h_k}$, $h_k \leq h_{k;0}$, where h_k is the maximum of the diameters of the elements in $\mathcal{T}_{h_k}$. We use Lagrangian conforming finite elements $S_p(\Omega_k, \mathcal{T}_{h_k})$ of order p on the individual subdomains and enforce homogeneous Dirichlet boundary conditions on $\partial\Omega \cap \partial\Omega_k$. We restrict ourselves to the geometrical conforming situation where the intersection between the boundary of any two different subdomains $\partial\Omega_l \cap \partial\Omega_k$, $k \neq l$, is either empty, a vertex, or a common edge. We only call it an interface in the latter case.

The mortar method is characterized by introducing Lagrange multiplier spaces given on the interfaces, which are used to "glue" the different parts $u|_{\Omega_k}$ of the weak solution u together. A suitable triangulation on the interfaces is necessary for the definition of a discrete Lagrange multiplier space. Each interface $\partial\Omega_l \cap \partial\Omega_k$ is associated with a one dimensional triangulation, inherited either from $\mathcal{T}_{h_k}$ or from $\mathcal{T}_{h_l}$. In general, these triangulations do not coincide. The interfaces are denoted by γ_m, $1 \leq m \leq M$. For each interface, there exists a couple $1 \leq l < k \leq K$ such that $\overline{\gamma}_m = \partial\Omega_l \cap \partial\Omega_k$. The one dimensional triangulation on γ_m is called Σ_m, and its elements are boundary edges of $\mathcal{T}_{h_l}$ or $\mathcal{T}_{h_k}$. The choice is arbitrary but fixed. Then, the non–mortar side is the one from which the Lagrange multiplier space inherits its triangulation. The adjacent side is called mortar side. In principle, independent triangulations for the Lagrange multiplier can be introduced at the interfaces. Then, we have to impose suitable conditions on the meshsize or add adequate bubble functions to the finite element space to obtain a discrete inf–sup condition. For stabilization techniques in case of the three–field approach, we refer to [BFMR98, BM99].

It is well known that the unconstrained product space

$$X_h := \prod_{k=1}^{K} S_1(\Omega_k, \mathcal{T}_{h_k})$$

equipped with the standard bilinear form is not suitable as a discretization of (2.1). Arbitrary jumps at the interfaces of the solution would be allowed. On the other hand, in general, it is not possible to construct a global continuous space with optimal approximation properties in case of non–matching meshes at the interfaces. To regain an approximation property of the discrete solution space, one has to impose suitable matching conditions at the interfaces. In [BMP93, BMP94], it has been shown that weak constraints across the interfaces are sufficient to guarantee approximation and consistency errors of order h if the weak solution u is smooth enough.

Two different ways of handling these constraints are common. The first one is to incorporate the constraints into the definition of the finite element space, resulting in a nonconforming formulation. The second equivalent approach is to represent the constraints by Lagrange multipliers and to consider a saddle point problem. A more detailed discussion of the advantages and disadvantages of these two approaches will be given in Section 3. Let us first consider the nonconforming formulation of the mortar method. For some suitable choice of the Lagrange multiplier space M_h, we define the constrained global finite element space V_h by

$$V_h := \{v \in X_h \mid \ b(v, \mu) = 0, \ \mu \in M_h\}\,, \tag{2.2}$$

where the bilinear form $b(\cdot,\cdot)$ is given as a duality pairing on the interfaces

$$b(v,\mu) := \sum_{m=1}^{M} \langle [v],\mu\rangle_{\gamma_m}, \quad v \in \prod_{k=1}^{K} H^1(\Omega_k),\ \mu \in \prod_{m=1}^{M} \left(H^{\frac{1}{2}}(\gamma_m)\right)'$$

and $[v]$ is the jump of v on γ_m. Here, $(H^{\frac{1}{2}}(\gamma_m))'$ denotes the dual space of $H^{\frac{1}{2}}(\gamma_m)$ and $\langle\cdot,\cdot\rangle_{\gamma_m}$ is the duality pairing between these two spaces. Of crucial importance is the suitable choice of M_h in (2.2),

$$\begin{aligned} M_h &:= \prod_{m=1}^{M} M_h(\gamma_m) \\ M_h(\gamma_m) &:= \operatorname{span}\{\psi_\ell \mid 1 \le \ell \le \nu_m := \dim W_{0;h}(\gamma_m)\}, \end{aligned}$$

where ψ_ℓ, $1 \le \ell \le \nu_m$ form a set of linear independent functions. Here, $W_h(\gamma_m)$ is the finite element trace space on γ_m and $W_{0;h}(\gamma_m) := W_h(\gamma_m) \cap H_0^1(\gamma_m)$. We note that also Lagrange multiplier spaces of smaller dimension can yield optimal results. Then if the basis functions of $M_h(\gamma_m)$ are chosen appropriately, the nonconforming mortar solution is obtained by: Find $u_h \in V_h$ such that

$$a(u_h,v) = (f,v)_0, \quad v \in V_h. \tag{2.3}$$

Here, the bilinear form $a(\cdot,\cdot)$ is defined by $a(v,w) := \sum_{k=1}^{K} \int_{\Omega_k} a\nabla v\, \nabla w + b\, v\, w\, dx$ for $v,w \in \prod_{k=1}^{K} H^1(\Omega_k)$. Associated with $a(\cdot,\cdot)$ is the energy norm $|\!|\!|\cdot|\!|\!|$, $|\!|\!|v|\!|\!|^2 := a(v,v)$. If $b \equiv 0$, the bilinear form $a(\cdot,\cdot)$ is not elliptic on the product space. However, it is elliptic on $Y \times Y$, where

$$Y := \{v \in \prod_{k=1}^{K} H^1(\Omega_k), \quad \int_{\gamma_m} [v]\, d\sigma = 0, \quad 1 \le m \le M\},$$

see [BM95]. Then, if the Lagrange multiplier space contains the constants on γ_m, V_h is a subspace of Y and thus the bilinear form $a(\cdot,\cdot)$ is uniformly elliptic on $Y \times Y$. Introducing the Lagrange multiplier as an additional unknown, we obtain an equivalent saddle point problem [Ben95]. In this approach, the space M_h explicitly plays the role of the Lagrange multiplier space. The weak continuity is not enforced on the space but guaranteed by the second equation of the saddle point problem: Find $(u_h,\lambda_h) \in X_h \times M_h$

$$\begin{aligned} a(u,v) + b(v,\lambda) &= (f,v)_0, & v \in X_h, \\ b(u,\mu) &= 0, & \mu \in M_h. \end{aligned} \tag{2.4}$$

In the rest of this section, we consider different Lagrange multiplier spaces for different discretizations yielding optimal results. In particular, we focus on dual basis spaces. Originally introduced for piecewise liner finite elements in 2D in [Woh98], this idea can be also realized for higher order finite elements or in 3D.

2.1. Lagrange multiplier spaces for the first order case in 2D. We consider four different Lagrange multiplier spaces for the piecewise linear case denoted by M_h^i, $1 \le i \le 4$, defining the constrained finite element spaces V_h^i. The corresponding mortar finite element solution of (2.3) and (2.4) is denoted by u_h^i and (u_h^i,λ_h^i), respectively. The nodal basis functions of $M_h^i(\gamma_m)$ are denoted by ψ_l^i, $1 \le l \le \nu_m$, where ν_m is the number of vertices x_l in the interior of γ_m. For simplicity, we suppress the index m in case of the basis functions and the vertices. The numeration of the vertices x_l is lexicographically, and the two endpoints of γ_m are denoted by x_0 and x_{ν_m+1}. Furthermore, the length h_l, $1 \le l \le \nu_m + 1$ is defined by $h_l := \|x_{l-1} - x_l\|$.

The shape of the different nodal basis functions is given in Figure 2.1.

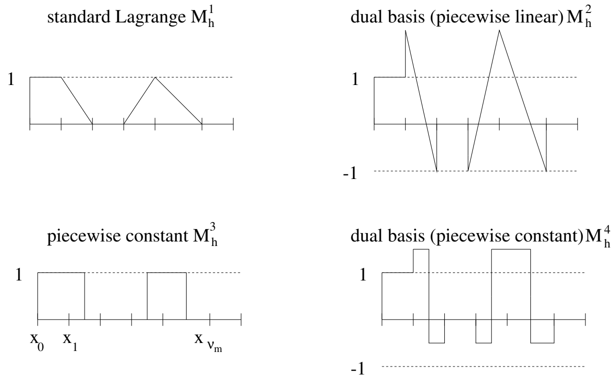


FIG. 2.1. *Nodal basis functions for* M_h^i, $1 \le i \le 4$

We set M_h^1 equal to the standard Lagrange multiplier space, see [BMP93, BMP94], and observe that the nodal basis functions are continuous and piecewise linear. The nodal basis functions in the interior of the interface are the standard hat functions, and only at the neighborhood of the two endpoints they are modified, see upper left part of Figure 2.1.

The second space M_h^2 has been originally introduced in [Woh98] for the mortar setting and is spanned by piecewise linear but discontinuous functions. The main advantage of this type of basis function is its biorthogonality to the standard hat functions. Here, we review the definition of the nodal basis functions

$$\psi_l^2(x) := \begin{cases} \frac{1}{h_l}(2\,\|x - x_{l-1}\| - \|x - x_l\|), & x \in [x_{l-1}, x_l], \\ \frac{1}{h_{l+1}}(2\,\|x - x_{l+1}\| - \|x - x_l\|), & x \in [x_l, x_{l+1}], \\ 0, & \text{elsewhere}, \end{cases} \qquad 2 \le l \le \nu_m - 1,$$

$$\psi_1^2(x) := \begin{cases} 1, & x \in [x_0, x_1), \\ \frac{1}{h_2}(2\,\|x - x_2\| - \|x - x_1\|), & x \in [x_1, x_2], \\ 0, & \text{elsewhere}, \end{cases}$$

and $\psi_{\nu_m}^2 := 1 - \sum_{l=1}^{\nu_m-1} \psi_l^2$. The shape functions are given in the upper right part of Figure 2.1.

Finally, the spaces M_h^3 and M_h^4 are based on piecewise constant functions, where M_h^4 is defined by dual basis functions as is M_h^2. Let us start with the definition of the nodal basis functions ψ_l^3 for $M_h^3(\gamma_m)$

$$\psi_l^3(x) := \begin{cases} 1, & x \in [\frac{1}{2}(x_{l-1} + x_l), \frac{1}{2}(x_l + x_{l+1})], \\ 0, & \text{elsewhere}, \end{cases} \qquad 2 \le l \le \nu_m - 1\,,$$

and $\psi_1^3(x)$ and $\psi_{\nu_m}^3(x)$ are equal one on $[x_0, \frac{1}{2}(x_1 + x_2)]$ and $[\frac{1}{2}(x_{\nu_m-1} + x_{\nu_m}), x_{\nu_m+1}]$, respectively, and zero elsewhere, see bottom left picture in Figure 2.1. Obviously, the space $M_h^3(\gamma_m)$ provides an order h approximation property in the L^2–norm.

Our last example for piecewise linear finite elements in 2D is a dual basis. In contrast to $M_h^2(\gamma_m)$, its basis functions are piecewise constant. We define for $2 \le l \le \nu_m - 1$

$$\psi_l^4(x) := \begin{cases} \frac{3}{2}, & x \in \left(\frac{1}{2}(x_{l-1}+x_l), \frac{1}{2}(x_l+x_{l+1})\right), \\ -\frac{1}{2}, & x \in [x_{l-1}, \frac{1}{2}(x_{l-1}+x_l)] \cup [\frac{1}{2}(x_l+x_{l+1}), x_{l+1}], \\ 0, & \text{elsewhere}, \end{cases}$$

and $\psi_1^4(x)$ and $\psi_{\nu_m-1}^4(x)$ are equal one on $[x_0, x_1]$ and $[x_{\nu_m}, x_{\nu_m+1}]$ and on $(x_1, x_2]$ and $[x_{\nu_m-1}, x_{\nu_m})$ they have the same structure as the other basis functions in the interior of the interface, respectively. The shape functions are shown in the lower right part of Figure 2.1. Then for $i = 2$ and $i = 4$, it is easy to verify that

$$\int_{\gamma_m} \psi_l^i \phi_k \, d\sigma = \delta_{lk} \int_{\gamma_m} \phi_k \, d\sigma, \quad 1 \le l, k \le \nu_m$$

where ϕ_k, $1 \le k \le \nu_m$ are the standard hat functions associated with the vertex x_k.

Optimal results are obtained for all these four Lagrange multiplier spaces in the L^2–norm, the energy norm $|||\cdot|||$ and a suitable norm $\|\cdot\|_M$ for the Lagrange multiplier

$$|||u - u_h^i|||^2 + \|\lambda - \lambda_h^i\|_M^2 \le C \sum_{k=1}^{K} h_k^2 \, |u|_{2;\,\Omega_k}^2 \tag{2.5}$$

$$\|u - u_h^i\|_0^2 \le C h^2 \sum_{k=1}^{K} h_k^2 \, |u|_{2;\,\Omega_k}^2 \, . \tag{2.6}$$

Here, we use standard Sobolev notations for the norms and seminorms and the constants $0 < c, C < \infty$ are generic ones not depending on the meshsize. The flux of the weak solution u is called λ and the norm $\|\cdot\|_M$ is either a $H_{00}^{1/2}$–dual norm or a mesh dependent L^2–norm

$$\|\mu\|_M^2 := \sum_{m=1}^{M} \|\mu\|^2_{(H_{00}^{1/2}(\gamma_m))'}, \quad \text{or} \quad \|\mu\|_M^2 := \sum_{m=1}^{M} \sum_{e \in \Sigma_m} h_e \|\mu\|_{0;e}^2 \, .$$

In case of (u_h^i, λ_h^i) and $i = 1$ or $i = 2$, the a priori estimates (2.5) and (2.6) are well known [Ben95, BMP93, BMP94, Woh97, Woh98]. For the cases $i = 3$ and $i = 4$, they can be found in [Woh99a]. We refer to [Woh98, Woh99a] for a numerical comparison of the discretization errors. It is shown that for examples including interior subdomains, singularities or discontinuous coefficients not only the asymptotic rates are the same but also that the quantitative errors are quite close.

Let us consider the structure of V_h^i in more detail. In particular, the values of $v \in V_h^i$ in the interior of the non–mortar side are given by its values on the mortar side and on the crosspoints. For $i = 1$ and $i = 3$, we find that the mass–matrix associated with the non–mortar sides is a tridiagonal matrix, whereas for $i = 2$ and $i = 4$, the mass–matrix is diagonal. The structure of the mass–matrix, or more precisely its inverse, determines the support of the basis functions of the constrained space V_h^i at the interfaces.

Figure 2.2 illustrates that the support of a basis function of V_h^1 or V_h^3 at the interface γ_m on the non–mortar side is a strip of length $|\gamma_m|$ and width h. Only in

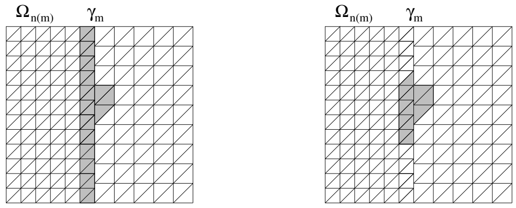


FIG. 2.2. *Support of a nodal basis function in V_h^1 or V_h^3 (left) and in V_h^2 or V_h^4 (right)*

case of $i = 2$ or $i = 4$, the locality of the support of the nodal basis functions is preserved.

In the rest of this section, we consider two other discretizations. The first one is based on the first order case in 3D whereas the second one is based on quadratic Lagrangian finite elements in 2D. For both cases a biorthogonal basis is introduced, and it defines the Lagrange multiplier space.

2.2. Lagrange multiplier spaces for the first order case in 3D. As second case, we consider first order Lagrangian finite elements in 3D. Two different situations are discussed separately, see Figure 2.3. In the first one, the triangulation on the interface γ_m is a tensor product grid whereas in the second one, it is a simplicial triangulation. For simplicity, we assume that the interfaces are rectangles.

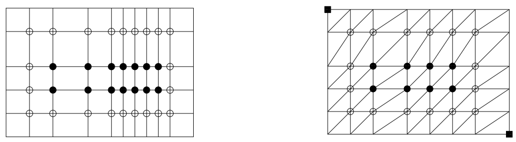


FIG. 2.3. *Trace of a hexahedral (left) and simplicial (right) triangulation on the interface*

As in the two dimensional case, the definition of the dual basis functions has to be modified in the neighborhood of the boundary $\partial\gamma_m$. We recall that $W_h(\gamma_m)$ is the trace space of bilinear or linear finite elements on γ_m and ϕ_l is the nodal basis function associated with the vertex x_l. Then in a first step, we define a basis biorthogonal to the standard nodal basis of $W_h(\gamma_m)$. In a second step, we reduce the number of dual basis functions by the number of vertices on the boundary of γ_m without loosing the biorthogonality and the approximation property of the dual space. We define $\widetilde{\psi}_l$ elementwise for $T \in \Sigma_m$ such that $\operatorname{supp} \widetilde{\psi}_l = \operatorname{supp} \phi_l$ and by

$$\widetilde{\psi}_{l_{|T}} := (4\phi_l - 2\phi_{a1(l)} - 2\phi_{a2(l)} + \phi_{o(l)})_{|T}\,, \quad T \in \operatorname{supp} \phi_l$$

in case of a hexahedral triangulation and by

$$\widetilde{\psi}_{l_{|T}} := (3\phi_l - \phi_{a1(l)} - \phi_{a2(l)})_{|T}, \quad T \in \operatorname{supp} \phi_l$$

in case of a simplicial triangulation. The index numeration $a1(l)$, $a2(l)$ and $o(l)$ is illustrated in Figure 2.4. The indices of the two adjacent vertices of x_l on ∂T are denoted by $a1(l)$ and $a2(l)$ and the index of the opposite one in case of a hexahedral triangulation by $o(l)$.

The basis functions ψ_l are associated with the vertices x_l being not on the boundary of γ_m. Now, we set $\psi_l := \widetilde{\psi}_l$ in the interior of γ_m, e.g. if $\partial(\operatorname{supp}\phi_l) \cap \partial\gamma_m = \emptyset$. The corresponding vertices are marked by a filled circle in Figure 2.3. For all other

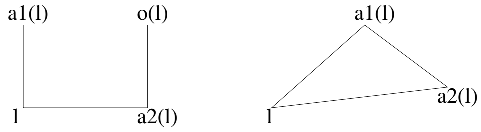


FIG. 2.4. *Index numeration in case of a rectangle (left) and a triangle (right)*

vertices $x_l \in \gamma_m$, marked by an empty circle in Figure 2.3, we have to modify the definition. To do so, we introduce two sets $\mathcal{M}$ and $\mathcal{M}_l$ given by

$$\mathcal{M}_l := \{x_j \mid x_j \in \partial\gamma_m \cap \partial(\mathrm{supp}\phi_l)\}\,, \quad \mathcal{M} := \{x_j \mid x_j \in \partial\gamma_m\} \setminus \bigcup_{x_l \in \gamma_m} \mathcal{M}_l\,.$$

We remark that $\mathcal{M}$ is always empty in case of a tensor grid. Only in case of simplicial triangulations, the set $\mathcal{M}$ can be non–empty. In the situation which is shown in the right part of Figure 2.3, it contains two elements associated with the upper left and the lower right vertex, marked with a filled square. Then, the modified dual basis functions are given by

$$\psi_l := \widetilde{\psi}_l + \sum_{x_j \in \mathcal{M}_l} \frac{1}{n_j} \widetilde{\psi}_j + \sum_{x_j \in \mathcal{M}} \alpha_{lj} \widetilde{\psi}_j\,. \tag{2.7}$$

Here, n_j is the number of sets $\mathcal{M}_k$ such that $x_j \in \mathcal{M}_k$ and $\alpha_{lj} = 1$ if $\partial(\mathrm{supp}\phi_l) \cap \partial(\mathrm{supp}\,\phi_j) \neq \emptyset$ and zero else. We refer to [BM97] for the introduction of a standard Lagrange multiplier space, and to [BD98] for some more sophisticated choices for the weights in (2.7). In particular, it is possible to replace $1/n_j$ by a weight involving the areas of the adjacent elements. It is easy to see that $\sum_{x_l \in \gamma_m} \psi_l = 1$ and thus $P_0(\gamma_m) \subset M_h(\gamma_m)$. Furthermore, a straightforward computation shows that the following biorthogonality relation holds

$$\int_{\gamma_m} \psi_l\, \phi_k\, dx = \delta_{lk} \int_{\gamma_m} \phi_k\, dx, \quad x_l, x_k \in \gamma_m\,.$$

As in the 2D case, the biorthogonality relation yields a diagonal mass–matrix interior of the non–mortar side, and the nodal basis functions of the constrained space have a local support. We remark that in the neighborhood of the boundary of γ_m, we have not $\mathrm{supp}\psi_l = \mathrm{supp}\,\phi_l$. We do not carry out the proof of optimal a priori bounds for the discretization errors and refer to [Woh99a] for more details.

2.3. Lagrange multiplier space for the second order case in 2D. The idea of using a biorthogonal basis with respect to the standard nodal finite element basis for the definition of the Lagrange multiplier space is not restricted to the first order case. Here, we will define a dual basis for piecewise quadratic Lagrangian finite elements. We replace in the definition of the unconstrained product space X_h the spaces $S_1(\Omega_k, \mathcal{T}_{h_k})$ by $S_2(\Omega_k, \mathcal{T}_{h_k})$, $1 \leq k \leq K$, and set for the rest of this subsection

$$X_h := \prod_{k=1}^{K} S_2(\Omega_k, \mathcal{T}_{h_k})\,.$$

To obtain an order h^2 a priori estimate for the discretization error in the energy norm, it is necessary that the Lagrange multiplier space $M_h(\gamma_m)$ contains the linear functions on the interface γ_m. The consistency error of the approach depends essentially on the approximation property of the Lagrange multiplier space. We refer to [SS98] in case of P_n–Lagrangian finite elements and conforming trace spaces for the Lagrange multiplier. For the approximation property, the $H_{00}^{1/2}$–stability of the mortar projection is relevant.

A possible choice for such a dual basis with local support is introduced in [Woh99a] and will be given in the following. The left part of Figure 2.5 shows the shape of the standard nodal basis functions of the trace space $W_{0;h}(\gamma_m)$. A corresponding dual basis is depicted in the right part of Figure 2.5. Again, the nodal basis functions of the dual basis are defined locally and are discontinuous.

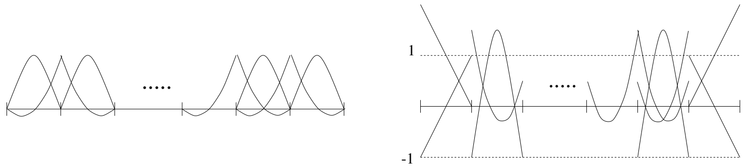


FIG. 2.5. *Nodal conforming basis (left) and nodal dual basis functions (right)*

The definition of the nodal dual basis functions depends on the index l, and we have to distinguish between the cases l even and l odd. Now, the dual basis functions are defined in terms of ϕ_l, $1 \le l \le \nu_m$ which are the standard nodal basis functions of $W_{0;h}(\gamma_m)$. We define ψ_l for $l = 2n+1$, $1 \le n \le (\nu_m - 3)/2$ by

$$\psi_l(x) := \begin{cases} -1 + \frac{5}{2}\phi_l(x), & x \in \operatorname{supp}\phi_l, \\ 0, & \text{elsewhere,} \end{cases}$$

and ψ_l for $l = 2n$, $2 \le n \le (\nu_m - 3)/2$ by

$$\psi_l(x) := \begin{cases} \frac{1}{2} - \frac{3}{4}\phi_{l-1}(x) + \phi_l(x) - \frac{3}{4}\phi_{l+1}(x), & x \in \operatorname{supp}\phi_l, \\ 0, & \text{elsewhere.} \end{cases}$$

The two first, ψ_1, ψ_2, and the two last, ψ_{ν_m-1}, ψ_{ν_m}, basis functions are defined separately

$$\begin{aligned} \psi_1(x) &:= \begin{cases} \frac{2}{h_1}\|x - x_1\|, & x \in \operatorname{supp}\phi_1 \\ 0, & \text{elsewhere,} \end{cases} \\ \psi_2(x) &:= \begin{cases} \frac{1}{h_1}(\|x - x_0\| - \|x - x_1\|), & x \in \operatorname{supp}\phi_1, \\ \frac{1}{2} + \phi_2(x) - \frac{3}{4}\phi_3(x), & x \in \operatorname{supp}\phi_2 \setminus \operatorname{supp}\phi_1, \\ 0, & \text{elsewhere,} \end{cases} \end{aligned}$$

ψ_{ν_m-1} and ψ_{ν_m} are given in the same way. Then, the biorthogonality relation between the basis functions ψ_l and ϕ_k can be shown by a straightforward computation.

3. Positive definite variational problem on the product space. In this section, we introduce an abstract framework for the definition of a new equivalent mortar setting based on the unconstrained product space. One possibility to work with the unconstrained product space, is discussed in [BH99, Ste98], where an idea introduced by Nitsche in [Nit70] is rediscovered and generalized to nonmatching triangulations. It turns out to be a penalty method where the penalty parameter does not depend on the meshsize and does not influence the condition number of the arising linear system. We refer to [BH99] for the details and some numerical results including a posteriori error estimators. Here, we carry out a different idea.

Very often the numerical solution of the mortar method is based on its saddle point formulation [AK95, AKP95, AMW96, AMW99, BD98, BDH99, BDL99, BDW98, EHI+97, EHI+98, HIK+98, Kuz95a, Kuz95b, Kuz98, KW95, Lac98, WW98, WW99]. Working with this indefinite formulation has the advantage that the unconstrained product spaces associated with a nested sequence of global triangulations are nested. In contrast, the nonconforming constrained spaces are not nested. Our approach is based on this observation, and we introduce a symmetric and positive definite variational problem on the unconstrained product space. It will be shown that the solution of this problem is nothing else than the mortar finite element solution. The idea is not restricted to first order discretizations or to 2D. We work with locally defined basis functions of $M_h(\gamma_m)$, satisfying $|\operatorname{supp}\psi_l| \leq C\,|\operatorname{supp}\phi_l|$ and $\|\psi_l\|_{0;\gamma_m} \leq C\,\|\phi_l\|_{0;\gamma_m}$. The only necessary condition is that the nodal basis functions of $M_h(\gamma_m)$ and $W_{0;h}(\gamma_m)$ form a biorthogonal set

$$\int\limits_{\gamma_m} \psi_l \phi_k \; d\sigma = c_l \delta_{lk} |\operatorname{supp}\phi_k|, \quad 1 \leq l,k \leq \nu_m \tag{3.1}$$

where $c \leq c_l \leq C$, $1 \leq l \leq \nu_m$. As in the previous section, we suppress the index m in case of the basis functions and also for the constants c_l. We further assume that the pair (X_h, M_h) defines a constrained space V_h such that the nonconforming variational problem (2.3) and the saddle point formulation (2.4) are well defined and yield optimal a priori bounds of the form (2.5) and (2.6) for the discretization errors. Examples for such pairings are given in the previous subsections. We recall that the only examples introduced in the previous subsections which do not satisfy the biorthogonality relation (3.1) are the standard Lagrange multiplier space M_h^1 and the space M_h^3. Only under the assumption (3.1), the Lagrange multiplier λ_h can be locally eliminated and easily evaluated in terms of the right hand side and the mortar solution u_h. This observation forms the starting point of our new approach.

The motivation for our approach is the following dilemma: Very often saddle point problems like (2.4) are solved by Uzawa–like algorithms, and inner and outer iteration schemes are obtained. In case of the mortar approach, recently such a multigrid technique has been investigated in [WW99]. In this case, a modified Schur complement system has to be solved iteratively within each smoothing step. This method is a generalization of [BD98, BDW98, WW98] where the modified Schur complement system has to be solved exactly. A cascadic conjugate gradient method investigated in [BDL99] is also based on the exact solution of a modified Schur complement system. Another approach is investigated in [Kuz95a, Kuz95b] and further used in [EHI+97, EHI+98, HIK+98, Kuz98]. It requires a good preconditioner for the exact Schur complement, and the iterative solution of the saddle point problem is obtained by a generalized Lanczos method. On the other hand, the positive definite formulation (2.3) is associated with a nonconforming space and standard multigrid techniques cannot be applied. We refer to [AMW96, AMW99, BDH99, CW96, Dry98a, Dry98b] for multilevel preconditioners and domain decomposition techniques. Recently, multigrid methods for the nonconforming formulation have been established [BDH99, GP98] involving the solution of a mass–matrix system in each smoothing step. Here, we introduce a third equivalent formulation. It is based on the unconstrained product space and gives rise to a symmetric positive definite formulation on which we can apply standard multigrid techniques. The rest of this section is devoted to the construction of a symmetric positive definite system living on the product space. Here, we will present the variational formulation on the product space as well as its algebraic form. The stiffness matrix associated with the unconstrained product space can be obtained by static condensation of the Lagrange multiplier. In case that the biorthogonality relation (3.1) holds, it can be easily evaluated in terms of the right hand side and the mortar solution u_h.

3.1. The variational formulation. At first glance, the saddle point problem (2.4) has the same structure for all pairings (X_h, M_h). However, there is an essential difference. Having the solution u_h of (2.3), the Lagrange multiplier λ_h can be obtained by solving a mass–matrix system on each interface. This is in general a global postprocessing step on the interface γ_m, and involves the inverse of the mass–matrix on the non–mortar side. In case that the biorthogonality (3.1) relation is satisfied, we find the coefficients $\lambda_{m;l}$ in

$$\lambda_{h_{|\gamma_m}} = \sum_{l=1}^{\nu_m} \lambda_{m;l}\psi_l$$

by a local postprocessing step. Then, the coefficients $\lambda_{m;l}$ are given in terms of u_h and f

$$\lambda_{m;l} = \frac{(f, \phi_l)_0 - a(u_h, \phi_l)}{\int\limits_{\gamma_m} \psi_l \phi_l \, d\sigma} ,$$

where ϕ_l is the nodal basis function of the trace space $W_{0;h}(\gamma_m)$ extended to the nodal basis function in X_h having its support in the subdomain $\Omega_{n(m)}$ associated with the non–mortar side. By means of linear functionals, the new formulation will be defined. We introduce $g_1 : X_h \longrightarrow M_h$ and $g_2 : X_h \longrightarrow X_h$. Both functionals are associated with the interfaces. The functional $g_1(v) := \sum_{m=1}^{M} \sum_{l=1}^{\nu_m} \beta_{m;l}\psi_l$ is uniquely defined if for each interface γ_m the coefficients $\beta_{m;l}$ are given. We set

$$\beta_{m;l} := \frac{v_{|\Omega_{n(m)}}(x_l)}{\int\limits_{\gamma_m} \psi_l \phi_l \, d\sigma} , \quad v \in X_h .$$

Let us recall that we have dropped the index m for the basis functions ψ_l for convenience. Furthermore, we define a linear functional $g_2(\cdot)$ being zero on V_h. It has the form $g_2(v) = \sum_{m=1}^{M} \sum_{l=1}^{\nu_m} \alpha_{m;l}\phi_l$ with

$$\alpha_{m;l} := \frac{b(v, \psi_l)}{\int\limits_{\gamma_m} \psi_l \phi_l \, d\sigma} .$$

Then, it is easy to see that $g_2(v)$ only lives in a small neighborhood of the non–mortar sides and is zero on the mortar sides of the interfaces. The kernel of $g_2(\cdot)$ is exactly V_h and $g_2(g_2(v)) = g_2(v)$. Now, there are two possibilities to show that an element $v \in X_h$ is also in V_h. The first one uses the definition (2.2) of V_h. It is sufficient to proof that $b(v, \psi_l) = 0$ for $1 \le l \le \nu_m$, $1 \le m \le M$. The second possibility is based on the projection $g_2(\cdot)$. For $v \in X_h$ to be in V_h, it is equivalent to satisfy $g_2(v) = 0$.

The following lemma shows the equivalence between the nonconforming mortar formulation (2.3) and a non–symmetric and a symmetric variational problem on the product space X_h. The idea behind the introduction of the new variational problem is to use the decomposition $v = (v - g_2(v)) + g_2(v)$, where $v - g_2(v) \in V_h$. We remark that the biorthogonality relation (3.1) is not necessary for $g_2(\cdot)$ to be well–defined. But if (3.1) does not hold, $g_2(\cdot)$ is no projection, and we do not find in general $v - g_2(v) \in V_h$.

LEMMA 3.1. *Let $u_h \in V_h$ be the unique solution of (2.3), then under the assumption (3.1), u_h is the unique solution of the non–symmetric variational problem*

$$a(u_h, v - g_2(v)) + b(u_h, g_1(v)) = (f, v - g_2(v))_0, \quad v \in X_h\,, \tag{3.2}$$

and of the positive definite symmetric variational problem

$$a(u_h - g_2(u_h), v - g_2(v)) + a(g_2(u_h), g_2(v)) = (f, v - g_2(v))_0, \quad v \in X_h\,. \tag{3.3}$$

Proof. In a first step, we show that the solution u_h of (2.3) satisfies (3.2) and (3.3). Using $u_h \in V_h$, we find $b(u_h, g_1(v)) = 0$ and $g_2(u_h) = 0$. Since $g_2(\cdot)$ is a projection, we get $g_2(v - g_2(v)) = 0$ and thus $v - g_2(v) \in V_h$. Combining these observations, we get immediately that u_h solves (3.2) and (3.3).

To obtain the unique solvability of (3.2) and (3.3), it is sufficient to show that each solution of (3.2) and (3.3), respectively, is an element of V_h and thus a solution of (2.3). An element $v \in X_h$ is in V_h if and only if $b(v, \mu) = 0$, $\mu \in M_h$ or equivalently if $g_2(v) = 0$. Let us assume that u_h satisfies (3.2). For each $1 \le m \le M$, $1 \le l \le \nu_m$, we have $g_2(\phi_l) = \phi_l$ and $g_1(\phi_l) = \alpha_l \psi_l$ and $\alpha_l \neq 0$. The choice $v := \phi_l$ yields

$$b(u_h, \psi_l) = 0\,,$$

and therefore u_h is the unique solution of (2.3). To proof $u_h \in V_h$ if u_h is the solution of (3.3), we use the second possibility and show $g_2(u_h) = 0$. Setting $v := g_2(u_h)$ yields

$$a(g_2(u_h), g_2(u_h)) = 0.$$

Finally, taking into account that the bilinear form $a(\cdot, \cdot)$ is coercive on $g_2(X_h)$, we find $g_2(u_h) = 0$.

Now, it is easy to see that the variational problem (3.3) is symmetric and positive definite. We define

$$\hat{a}_h(v, w) := a(v - g_2(v), w - g_2(w)) + a(g_2(v), g_2(w))\,, \qquad v, w \in X_h\,. \tag{3.4}$$

Then $\hat{a}_h(\cdot, \cdot)$ is a symmetric bilinear form satisfying $\hat{a}_h(v, v) \ge 0$. For each $v \in X_h$ satisfying $\hat{a}_h(v, v) = 0$, we find $g_2(v) = 0$ and $a(v, v) = 0$. Taking into account that the kernel of $g_2(\cdot)$ is V_h and that $a(\cdot, \cdot)$ is V_h–elliptic, we get $v = 0$. □

REMARK 3.2. *We remark that the idea of dual basis functions is not restricted to the coupling of conforming Lagrangian finite elements. In [WW98], the coupling between mixed Raviart–Thomas finite elements and conforming piecewise linear Lagrangian finite elements is realized. Due to the different role of Dirichlet and Neumann boundary conditions a saddle point formulation without Lagrange multipliers on the interfaces can be established. Using the equivalent nonconforming approach with Crouzeix–Raviart elements instead of the mixed Raviart–Thomas elements, a piecewise constant Lagrange multiplier on the interfaces enters the formulation. Taking the definition of the degrees of freedom for the Crouzeix–Raviart elements into account, it is easy to see that we have also in this setting the required biorthogonality relation between the nodal basis functions of the trace space and the ones of the Lagrange multiplier space.*

For the analysis of the multigrid method, we assume that the global triangulation is quasi uniform. In the rest of this section, we consider the bilinear form $\hat{a}_h(\cdot,\cdot)$ in more detail. We focus on the continuity and the ellipticity of $\hat{a}_h(\cdot,\cdot)$. The continuity of $\hat{a}_h(\cdot,\cdot)$ is closely related to the stability of the decomposition of $v \in X_h$ in $v - g_2(v)$ and $g_2(v)$. This decomposition is not uniformly stable in h with respect to the energy norm. Taking an element $v \in X_h$ being piecewise constant on the subdomains, we find that

$$\hat{a}_h(v,v) \le C \left(\|v\|_0^2 + \frac{1}{h} \sum_{m=1}^{M} \|[v]\|_{0;\gamma_m}^2 \right) .$$

In the following $\|\cdot\|$ stands for the broken H^1–norm on Ω. More general, we obtain the following upper bound for $\hat{a}_h(v,v)$, $v \in X_h$

$$\hat{a}_h(v,v) \le 2(a(v,v) + a(g_2(v), g_2(v))) \le C \left(\|v\|_1^2 + \frac{1}{h} \sum_{m=1}^{M} \|[v]\|_{0;\gamma_m}^2 \right) \le \frac{C}{h^2} \|v\|_0^2 .$$

A lower bound can be easily obtained by the uniform ellipticity of $a(\cdot,\cdot)$ on $V_h \times V_h$ and on $g_2(X_h) \times g_2(X_h)$

$$\hat{a}_h(v,v) \ge C \left(\|v - g_2(v)\|_1^2 + \|g_2(v)\|_1^2 \right) \ge C \|v\|_1^2 .$$

Combining the upper and lower bounds for the bilinear form $\hat{a}_h(\cdot,\cdot)$, we get that the condition number of the associated stiffness–matrix is bounded by C/h^2 as in the standard conforming finite element setting.

3.2. The algebraic formulation. In this subsection, we consider the algebraic formulations of the non–symmetric (3.2) and of the symmetric (3.3) variational problem. The starting point is the algebraic formulation of the saddle point problem (2.4)

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} . \tag{3.5}$$

Here, we use the same notation for the vector presentation of the solution and the solution as an element in X_h. The matrix A is the stiffness–matrix associated with the bilinear form $a(\cdot,\cdot)$ on $X_h \times X_h$ and the matrices B and B^T are associated with the bilinear form $b(\cdot,\cdot)$ on $X_h \times M_h$. To see the consequences of the biorthogonality relation (3.1), we have to consider the matrix B in more detail. We decompose u_h into three components $u_h^T = (u_I^T, u_M^T, u_N^T)$. The first one u_I stands for all degrees of freedom associated with the interior nodes of the subdomains. The second one u_M includes all degrees of freedom on the mortar sides and on the boundary of the non–mortar sides. Finally, the third one u_N stands for all degrees of freedom associated with the interior nodes on the non–mortar sides. We note that each degree of freedom is contained in exactly one of the three groups. By means of this decomposition, we can rewrite the saddle point problem (3.5) and find

$$\begin{pmatrix} A_{II} & A_{IM} & A_{IN} & 0 \\ A_{MI} & A_{MM} & A_{MN} & M \\ A_{NI} & A_{NM} & A_{NN} & D \\ 0 & M^T & D & 0 \end{pmatrix} \begin{pmatrix} u_I \\ u_M \\ u_N \\ \lambda_h \end{pmatrix} = \begin{pmatrix} f_I \\ f_M \\ f_N \\ 0 \end{pmatrix} . \tag{3.6}$$

The matrix M is a block mass–matrix and the band width depends on the local ratio of the meshsizes on mortar and adjacent non–mortar sides. The biorthogonality relation (3.1) guarantees that D is a diagonal matrix with positive entries. The third line of the saddle point problem (3.6) yields

$$\lambda_h = D^{-1}(f_N - A_{NI}u_I - A_{NM}u_M - A_{NN}u_N)\,.$$

Introducing $W^T := (0, 0, D^{-1})$, we can rewrite λ_h in terms of W^T and the defect on the product space

$$\lambda_h = W^T(f - Au_h)\,. \tag{3.7}$$

Using this elimination in (3.5), we find

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ -W^T A \end{pmatrix} u_h = \begin{pmatrix} (\mathrm{Id} - BW^T)f \\ 0 \end{pmatrix}. \tag{3.8}$$

In a last step, we reduce the number of equations by the dimension of M_h. This can be done in different ways. We multiply (3.8) from the left by (v^T, μ^T) and set μ as a function of f and v. One possible choice is

$$\mu := W^T v$$

yielding the algebraic non–symmetric system

$$(\mathrm{Id}, W) \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ -W^T A \end{pmatrix} u_h = (\mathrm{Id} - BW^T)f. \tag{3.9}$$

The choice

$$\mu := W^T A(WB^T - \mathrm{Id})v$$

gives rise to a similar but symmetric positive definite system

$$(\mathrm{Id}, (BW^T - \mathrm{Id})AW) \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ W^T A(WB^T - \mathrm{Id}) \end{pmatrix} u_h = (\mathrm{Id} - BW^T)f. \tag{3.10}$$

Here, we have replaced (3.7) by the equivalent form $\lambda_h = W^T(f + A(WB^T - \mathrm{Id})u_h)$. Now, the following lemma shows the relation between the algebraic formulations (3.9) and (3.10) and the variational equations (3.2) and (3.3).

LEMMA 3.3. *The algebraic form of the variational problems (3.2) and (3.3) is given by (3.9) and (3.10), respectively.*

Proof. To start, we rewrite the two linear functions $g_1(\cdot)$ and $g_2(\cdot)$ in an algebraic form. Then, $g_1(v)$ and $g_2(v)$ are given by

$$W^T v \quad \text{and} \quad WB^T v\,,$$

respectively. We apply B^T on $v \in X_h$ and we find $WB^T v = 0$ if and only if $v \in V_h$. The multiplication with W can be interpreted as a scaled mapping from M_h onto X_h such that the result $WB^T v$ lives in a small strip of width h in the neighborhood of the non–mortar side. Using this observation in the variational form (3.2), we get its algebraic formulation

$$(\mathrm{Id} - BW^T)Au_h + WB^T u_h = (\mathrm{Id} - BW^T)f\,.$$

Comparing this system with (3.9), a straightforward computation gives the equality. The second assertion can be as easily established. Starting with the variational formulation (3.3), we obtain for its algebraic form

$$(\mathrm{Id} - BW^T)A(\mathrm{Id} - BW^T)^T u_h + BW^T AWB^T u_h = (\mathrm{Id} - BW^T)f\,,$$

which is nothing else than (3.10). □

REMARK 3.4. *The numerical realization of our multigrid method will be based on the non–symmetric stiffness matrix* $A_{\mathrm{num}} :=$

$$\begin{pmatrix} A_{II} & A_{IM} - A_{IN}D^{-1}M^T & 0 \\ A_{MI} - MD^{-1}A_{NI} & \substack{A_{MM}+2MD^{-1}A_{NN}D^{-1}M^T \\ -MD^{-1}A_{NM}-A_{MN}D^{-1}M^T} & MD^{-1}A_{NN} \\ 0 & D^{-1}M^T & \mathrm{Id} \end{pmatrix}. \tag{3.11}$$

This non–symmetric system is obtained from (3.10) by multiplying the third line with A_{NN}^{-1}. *The analysis of the method will be carried out for the symmetric formulation (3.10). A special class of smoothers allows to work with two different formulations at the same time.*

4. Multigrid convergence. The main difficulty in solving the linear systems arising from mortar discretizations is the handling of the constraints at the interfaces. Using the approach presented in this paper, standard multigrid methods can be applied directly to the positive definite formulation (3.3). In contrast to the constrained nonconforming case, the unconstrained product spaces associated with a nested sequence of triangulations are nested. However, using the standard injection to define the restriction and prolongation does not provide a suitable approximation property. We observe that the bilinear form $\hat{a}_h(\cdot,\cdot)$ defined in (3.4) on $X_h \times X_h$ is mesh dependent due to the use of the linear form $g_2(\cdot)$. Let us recall that $\hat{a}_h(v,v)$ is bounded by the mesh dependent norm $|\!|\!|v|\!|\!|^2 + 1/h\,\|[v]\|_0^2$. Working with the bilinear form $\hat{a}_h(\cdot,\cdot)$ on $X_H \times X_H$, $X_H \subset X_h$, $H > h$, we cannot expect an approximation property where the constant does not depend on the ratio H/h. To overcome this difficulty, we define level dependent bilinear forms in Subsection 4.1. By means of the a priori estimates and the definition of the level dependent bilinear form, we establish an approximation property in Subsection 4.2. Finally in Subsection 4.3, we consider a general class of smoothing operators. We derive the smoothing property and the stability of the smoother in suitable norms, thus completing the necessary ingredients for the proof of the $\mathcal{W}$–cycle convergence.

4.1. Bilinear forms. For convenience, let us denote the finite element spaces X_{h_l} associated with a nested sequence of triangulations with meshsize parameter h_l on level l by X_l, $l = 1, \ldots, L$ and let us assume $2\,h_{l+1} = h_l$. The finest level L is arbitrary, but fixed. We set $\dim X_l =: n_l$ and denote the Euclidian scalar product in $\mathbb{R}^{n_l}$ by $(\cdot,\cdot)$. Furthermore, we use the same notation for $v_l \in X_l$ and its vector representation $v_l \in \mathbb{R}^{n_l}$ with respect to the standard nodal basis. This scaling yields that the L^2–norm of $v_l \in X_l$ is equivalent to the Euclidian norm $\|v_l\| := (v_l, v_l)^{1/2}$ of $v_l \in \mathbb{R}^{n_l}$ scaled by h_l

$$c\,\|v_l\|_0 \;\le\; h_l\,\|v_l\| \;\le\; C\,\|v_l\|_0\,.$$

Observing that the definition of the linear forms $g_1(\cdot)$ and $g_2(\cdot)$ depends on the level l, we focus on the introduction of a level dependent bilinear form and a suitable transfer operator. To expose this dependency, we provide an additional index for the projections $g_2^l(\cdot)$ and write

$$\hat{a}_l(u_l, v_l) = a(u_l - g_2^l(u_l), v_l - g_2^l(v_l)) + a(g_2^l(u_l), g_2^l(v_l)), \qquad u_l, v_l \in X_l . \tag{4.1}$$

Using this notation and denoting the stiffness matrix and right hand side defined in (3.10) by $A_{l;\mathrm{sym}}$ and $f_{l;\mathrm{sym}}$, we can rewrite the linear system (3.10) on level l as

$$A_{l;\mathrm{sym}} u_l = f_{l;\mathrm{sym}} ,$$

which is equivalent to the variational equation

$$\hat{a}_l(u_l, v_l) = (f, v_l)_0 , \qquad v_l \in X_l .$$

Now, we have to specify the bilinear forms on the coarse grid spaces. To do so, let us first denote the standard grid transfer operators with respect to the nodal basis by $I_l^{l-1} \colon X_l \longrightarrow X_{l-1}$ and $I_{l-1}^l \colon X_{l-1} \longrightarrow X_l$. As usual, I_{l-1}^l is chosen to be the natural injection and the restriction is defined by $(I_l^{l-1})^T := I_{l-1}^l$. Thus, it holds

$$(I_l^{l-1} v_l, w_{l-1}) = (v_l, I_{l-1}^l w_{l-1}), \qquad v_l \in X_l , \; w_{l-1} \in X_{l-1} .$$

At first glance, it might be possible to use the restriction of $\hat{a}_L(\cdot,\cdot)$ onto X_l, i.e. using

$$I_{l+1}^l \cdots I_L^{L-1} A_{L;\mathrm{sym}} I_{L-1}^L \cdots I_l^{l+1} .$$

Unfortunately, in this case no suitable approximation property can be established. Thus, we equip the space $X_l \times X_l$ with the level dependent bilinear form $\hat{a}_l(\cdot,\cdot)$ given by (4.1). In the next subsection, it is shown that for the finite element spaces X_l equipped with the level dependent bilinear forms $\hat{a}_l(\cdot,\cdot)$ the approximation property holds.

In addition to define our multigrid method, we need a defect restriction. Let $\tilde{w}_{l-1}$ be the solution of the defect problem in X_{l-1} associated with $\hat{a}_{l-1}(\cdot,\cdot)$ and $\tilde{d}_{l-1} := I_l^{l-1} d_l \in X_{l-1}$. Then, even if $W_l^T d_l = 0$, in general $W_{l-1}^T \tilde{d}_{l-1} \neq 0$. As a consequence, the constraints are in general not satisfied on level $l-1$, i.e. $B_{l-1}^T \tilde{w}_{l-1} \neq 0$. This observation motivates the definition of a modified restriction operator $(\widetilde{I}_{\mathrm{mod}})_l^{l-1}$ satisfying $W_{l-1}^T d_{l-1} = 0$ if $W_l^T d_l = 0$. Our new restriction operator is defined by

$$(\widetilde{I}_{\mathrm{mod}})_l^{l-1} d_l := (\mathrm{Id} - B_{l-1} W_{l-1}^T)\, I_l^{l-1}\, (\mathrm{Id} - B_l W_l^T)\, d_l \; + \; B_{l-1} W_{l-1}^T I_l^{l-1} B_l W_l^T\, d_l .$$

We remark that the restriction operator is based on the decomposition $d_l = (\mathrm{Id} - B_l W_l^T)\, d_l + B_l W_l^T\, d_l$. Now, the coarse grid correction w_{l-1} is given as the solution of the defect equation

$$A_{l-1;\mathrm{sym}} = d_{l-1} , \tag{4.2}$$

where $d_{l-1} := (\widetilde{I}_{\mathrm{mod}})_l^{l-1} d_l$. Thus if the iterates on level l satisfy the constraints, the correction w_{l-1} satisfies the constraints on level $l-1$, i.e. $B_{l-1}^T w_{l-1} = 0$.

In the following remark, we discuss how the matrices $A_{l;\mathrm{sym}}$ for $l < L$ are assembled.

REMARK 4.1. *Firstly, we obtain the matrices A_{l-1}, $l \leq L$, by recursive Galerkin assembling $A_{l-1} = I_l^{l-1}\, A_l I_{l-1}^l$. These matrices correspond to the bilinear form $a(\cdot,\cdot)$ given on the unconstrained product space, i.e.*

$$(A_l u_l, v_l) = a(u_l, v_l), \qquad u_l, v_l \in X_l .$$

Using these matrices, the modified matrices $A_{l;\mathrm{num}}$ *can be computed by local modifications on each level.*

In a second remark, we consider the modified restriction operator in more detail. Working on a suitable subspace, the definition of the restriction can be considerably simplified.

REMARK 4.2. *In case that the defect* d_l *on level* l *satisfies* $W_l^T d_l = 0$, *we can compute* d_{l-1} *by* $d_{l-1} = (I_{\mathrm{mod}})_l^{l-1} d_l$, *where* $(I_{\mathrm{mod}})_l^{l-1}$ *is the simplified restriction operator*

$$(I_{\mathrm{mod}})_l^{l-1} := (\mathrm{Id} - B_{l-1} W_{l-1}^T)\, I_l^{l-1} \,. \tag{4.3}$$

As we will see in Section 4.3, our special class of smoothers satisfies $W_l^T d_l = 0$ *by construction.*

4.2. Approximation property. One essential tool to establish level independent convergence rates for the $\mathcal{W}$–cycle is a suitable approximation property. The saddle point problem (2.4) forms the starting point of our analysis. Based on the a priori estimates for the saddle point problem (2.4), we establish an approximation property in the L^2–norm. In [Woh99b], the approximation property for the saddle point problem on level l is shown if the right hand side is orthogonal to the space $X_{l-1} \times M_{l-1}$ with respect to a level dependent bilinear form. The result is established for the standard Lagrange multiplier spaces which are nested. We cannot apply this result to our setting since the Lagrange multiplier spaces have not to be nested. In particular, the examples satisfying (3.1) presented in Section 2 are non–nested. Here, we establish a weaker result for the more general situation of non–nested Lagrange multiplier spaces.

For the rest of this section, we assume full H^2–regularity of the problem. Let w_l and w_{l-1} be the solution of $A_{l;\mathrm{sym}} w_l = d_l$ and $A_{l-1;\mathrm{sym}} w_{l-1} = d_{l-1}$, respectively, where the stiffness matrices and d_{l-1} are defined in the previous subsection. The following lemma gives an upper bound for $w_l - I_{l-1}^l w_{l-1}$ in the Euclidian vector norm.

LEMMA 4.3. *Under the assumption that* $W_l^T d_l = 0$, *the following upper bound for* $w_l - I_{l-1}^l w_{l-1}$ *holds*

$$\| w_l - I_{l-1}^l w_{l-1} \| \leq C \, \| d_l \|$$

with a constant C *independent of the refinement level* l.

Proof. Observing that $W_{l-1}^T d_{l-1} = W_{l-1}^T (\mathrm{Id} - B_{l-1} W_{l-1}^T) I_l^{l-1} d_l = 0$, we find $B_l^T w_l = B_{l-1}^T w_{l-1} = 0$. To proof the assertion, we consider a corresponding saddle point problem. Defining $\lambda_l := W_l^T (d_l - A_l w_l)$ and $\lambda_{l-1} := W_{l-1}^T (I_l^{l-1} d_l - A_{l-1} w_{l-1})$, we find that (w_l, λ_l) and (w_{l-1}, λ_{l-1}) solve the saddle point problem

$$\begin{pmatrix} A_l & B_l \\ B_l^T & 0 \end{pmatrix} \begin{pmatrix} w_l \\ \lambda_l \end{pmatrix} = \begin{pmatrix} d_l \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_{l-1} & B_{l-1} \\ B_{l-1}^T & 0 \end{pmatrix} \begin{pmatrix} w_{l-1} \\ \lambda_{l-1} \end{pmatrix} = \begin{pmatrix} I_l^{l-1} d_l \\ 0 \end{pmatrix},$$

respectively. Associated with d_l is a unique $f_d \in X_l \subset L^2(\Omega)$ such that for $v \in X_l$

$$(f_d, v)_0 = (d_l, v) \,,$$

and $h_l \, \| f_d \|_0 \leq C \, \| d_l \|$. Now, we define the continuous variational problem by: Find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = (f_d, v)_0 , \quad v \in H_0^1(\Omega) \,.$$

Then, w_l and w_{l-1} are the corresponding nonconforming mortar finite element solutions of this variational problem on level l and $l-1$, respectively. Using the full H^2–regularity and the a priori estimates in the L^2–norm, we obtain

$$\begin{aligned} \|w_l - I_{l-1}^l w_{l-1}\| &\le \tfrac{C}{h_l}\|w_l - w_{l-1}\|_0 \le \tfrac{C}{h_l}(\|w_l - w\|_0 + \|w - w_{l-1}\|_0) \\ &\le Ch_l\|w\|_2 \le Ch_l\|f_d\|_0 \le C\|d_l\|. \end{aligned}$$

□

In case that the Lagrange multiplier spaces are non–nested, $M_{l-1} \not\subset M_l$, we can not write $w_l - w_{l-1} \in X_l$ as the solution of a saddle point problem on $X_l \times M_l$. Then, the approximation property has to be established in terms of $\|w_l - w\|$ and $\|w_{l-1} - w\|$ where w is a suitable element in $H_0^1(\Omega)$. We point out that the approximation property is only shown for the special case that $W_l^T d_l = 0$. Thus, we have to use special types of smoothers satisfying this condition.

4.3. Smoothing property. In addition to the approximation property, a suitable smoothing property has to be established to obtain level independent convergence rates. To measure the smoothing effect, the iteration error $\|e^\nu\|$ in the $\|A_{\text{sym}} \cdot \|$–norm has to be bounded by the $\|\cdot\|$–norm of the start iterate e^0. Note, that the discrete norm $\|A_{\text{sym}} \cdot \|$ is not equivalent to the H^2–norm. In contrast to the two grid convergence analysis, approximation and smoothing property are not sufficient to obtain a level independent convergence rate in the full multigrid case. Additionally, the stability constant of the smoothing operator in the $\|\cdot\|$–norm enters the estimates.

In this subsection, we construct suitable smoothing operators for the system on the unconstrained product space. We suppress all indices indicating the actual level. In particular, we focus on a special structure of the smoother yielding iterates which satisfy the constraints at the interface. As a consequence, we obtain after ν smoothing steps, $\nu \ge 1$, a defect vector satisfying $W^T d = 0$. Only in this case, we are in the setting of Lemma 4.3, which states the approximation property. To define a suitable smoother for our multigrid method, we start with a 2×2 block decomposition of A_{sym}. The first block is obtained by grouping u_I and u_M together. Introducing $u_S^T := (u_I^T, u_M^T)$, (3.10) can be rewritten as

$$\begin{pmatrix} A_{SS} & A_{SN} \\ A_{NS} & A_{NN} \end{pmatrix} \begin{pmatrix} u_S \\ u_N \end{pmatrix} = \begin{pmatrix} f_S \\ 0 \end{pmatrix},$$

where $f_S^T := ((f_{\text{sym}})_I^T, (f_{\text{sym}})_M^T)$. The blocks A_{SS} and $A_{SN} = A_{NS}^T$ are given by

$$A_{SS} := \begin{pmatrix} (A_{\text{sym}})_{II} & (A_{\text{sym}})_{IM} \\ (A_{\text{sym}})_{MI} & (A_{\text{sym}})_{MM} \end{pmatrix}, \qquad A_{SN} := \begin{pmatrix} (A_{\text{sym}})_{IN} \\ (A_{\text{sym}})_{MN} \end{pmatrix},$$

where the block matrices are submatrices of A_{sym}. The matrix A_{SS} is symmetric positive definite, the eigenvalues of which are bounded from below by ch^2 and from above by a constant C. A_{NN} is the symmetric positive sub block of A associated with the interior degrees of freedom on the non–mortar sides. Its eigenvalues are bounded by ch and C. Let us assume that G_S defines a symmetric and positive definite smoothing operator for A_{SS}, satisfying

$$\sigma(G_S^{-1} A_{SS}) \subset [0,1], \quad c\,\text{Id} \le G_S \le C\,\text{Id}. \tag{4.4}$$

Here, $\sigma(T)$ denotes the spectrum of the operator T. We define our iteration matrix G^{-1} by

$$G^{-1} := \begin{pmatrix} G_S^{-1} & -G_S^{-1} B_{SN} D^{-1} \\ -D^{-1} B_{NS} G_S^{-1} & \frac{3}{2} D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} + \alpha \mathrm{Id} \end{pmatrix}, \tag{4.5}$$

where $B_{NS} = (0, M^T)$ and $B_{SN} = B_{NS}^T$. Here, M is the mass–matrix defined in the previous section. The coefficient α is defined in terms of the maximal eigenvalue $\lambda_{\max}$ of A_{NN} satisfying $c \le \lambda_{\max} \le C$, $\alpha := 0.5 \cdot \lambda_{\max}^{-1}$.

By means of G^{-1}, we define our iteration scheme. The ν–th iterate $y^\nu \in X_h$ is given by

$$y^\nu = y^{\nu-1} + G^{-1}\left(d - A_{\mathrm{sym}}\, y^{\nu-1}\right), \quad \nu \ge 1, \tag{4.6}$$

where y is the solution of $A_{\mathrm{sym}}\, y = d$ and $y^0 \in X_h$ is a given start iterate. The stability constant of the iteration (4.6) depends essentially on the condition number of the operator G^{-1}. The following lemma shows that the algebraic properties of G^{-1} are inherited from G_S^{-1}. In particular, the condition number of G^{-1} is bounded by the one of G_S.

LEMMA 4.4. *Under the assumptions (4.4), G^{-1} defined in (4.5) is a symmetric and positive definite operator satisfying*

$$c\,\mathrm{Id} \;\le\; G \;\le\; C\mathrm{Id}$$

with constants c, C not depending on the meshsize.

Proof. The symmetry of G^{-1} is a consequence of (4.5). It remains to estimate the eigenvalues of G^{-1}. A straightforward computation shows that G^{-1} is spectrally equivalent to its block diagonal matrix

$$c_1 N \le x^T \begin{pmatrix} G_S^{-1} & -G_S^{-1} B_{SN} D^{-1} \\ -D^{-1} B_{NS} G_S^{-1} & \frac{3}{2} D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} + \alpha \mathrm{Id} \end{pmatrix} x \le C_1 N\,,$$

where $N := \left(x_S^T G_S^{-1} x_S + x_N^T D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} x_N + \alpha x_N^T x_N\right)$, $x^T := (x_S^T, x_N^T)$, and $c_1 := (5 - \sqrt{17})/4$, $C_1 := (5 + \sqrt{17})/4$. Finally, we consider N in more detail and find by means of $\|B_{SN} D^{-1}\| \le C$ the following upper bound

$$N \;\le\; C\,\|G_S^{-1}\|\, \left(\|x_S\|^2 + \|B_{SN} D^{-1} x_N\|^2 + \alpha \|x_N\|^2\right) \;\le\; C\,\|G_S^{-1}\|\,\|x\|^2\,.$$

In the first inequality, we have used the fact that $\|G_S\| \le C$ and in the last one α is bounded. On the other hand observing $\alpha \ge c$ and (4.4), a lower bound for N can be given in terms of $\|G_S\|$

$$N \;\ge\; \|G_S\|^{-1}\,\left(\|x_S\|^2 \,+\, \|B_{SN} D^{-1} x_N\|^2 \,+\, \alpha\,\|x_N\|^2\right) \;\ge C\,\|G_S\|^{-1}\,\|x\|^2\,.$$

We remark that in general an inequality of the form $\|B_{SN} D^{-1} x_N\| \ge c\,\|x_N\|$ with $c > 0$ does not hold. Furthermore, it is easy to see that the condition number of G is bounded by the condition number of G_S. Then, the assumption (4.4) yields the assertion. □

Having the positive definiteness of G^{-1}, to obtain a smoother, we still have to show that G provides a good approximation of the high frequency part of A_{sym}. We recall that the modified defect restriction introduced in Subsection 4.1 guarantees that the right hand side d of the defect equation has the form $d = (d_S^T, 0)^T$, if the iterate on the previous level satisfies the constraints.

LEMMA 4.5. *Under the assumptions $B^T y^0 = 0$ and 4.4 on G_S, we obtain*

$$\|A_{\mathrm{sym}}\, e^\nu\| \le \frac{C}{\nu}\, \|e^0\|\,, \qquad \nu \ge 1,$$

where $e^\nu := y - y^\nu$. Furthermore, each iterate y^ν satisfies $B^T y^\nu = 0$.

Proof. We start with the proof of $B^T y^\nu = 0$ by induction. Assuming $B^T y^\nu = 0$, we find for $B^T y^{\nu+1}$

$$B^T y^{\nu+1} = B^T y^\nu + B^T G^{-1} A_{\mathrm{sym}}\, e^\nu = B^T G^{-1} A_{\mathrm{sym}}\, e^\nu\,.$$

In a next step, we consider $B^T G^{-1} A_{\mathrm{sym}}$ in more detail. Observing $A_{NN} D^{-1} B^T y = 0$ yields $B^T y = 0$, and we get

$$\begin{aligned} B^T G^{-1} A_{\mathrm{sym}}\, e^\nu &= B^T G^{-1} \begin{pmatrix} A_{SS} & A_{SN} \\ A_{NS} & A_{NN} \end{pmatrix} \begin{pmatrix} e_S^\nu \\ e_N^\nu \end{pmatrix} \\ &= B^T \begin{pmatrix} G_S^{-1} & -G_S^{-1} B_{SN} D^{-1} \\ -D^{-1} B_{NS} G_S^{-1} & \frac{3}{2} D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} + \alpha \mathrm{Id} \end{pmatrix} \begin{pmatrix} A_{SS} e_S^\nu + A_{SN} e_N^\nu \\ 0 \end{pmatrix} \\ &= B^T \begin{pmatrix} \mathrm{Id} \\ -D^{-1} B_{NS} \end{pmatrix} G_S^{-1} (A_{SS} e_S^\nu + A_{SN} e_N^\nu) \\ &= \left(B_{NS} - D D^{-1} B_{NS}\right) G_S^{-1} \left(A_{SS} e_S^\nu + A_{SN} e_N^\nu\right) = 0\,. \end{aligned}$$

Here, we have used $A_{NS} = A_{NN} D^{-1} B_{NS}$. Now, the assumption $B^T y^0 = 0$ yields the assertion $B^T y^{\nu+1} = 0$ for $\nu \ge 1$.

In a second step, we proof the smoothing property of iteration (4.6). As shown in Lemma 4.4, the operator G^{-1} is symmetric and positive definite. Then, $G^{-1/2}$ is well defined and $A_{\mathrm{sym}} e^\nu$ can be rewritten as

$$A_{\mathrm{sym}} e^\nu = G^{1/2} (G^{-1/2} A_{\mathrm{sym}} G^{-1/2}) (\mathrm{Id} - G^{-1/2} A_{\mathrm{sym}} G^{-1/2})^\nu G^{1/2} e^0\,.$$

Having in mind that the norm of a symmetric matrix is bounded by its spectral radius, it is sufficient to estimate the spectrum of $G^{-1/2} A_{\mathrm{sym}} G^{-1/2}$

$$\|A_{\mathrm{sym}} e^\nu\| \le \|G\| \max_{x \in \sigma(G^{-1} A_{\mathrm{sym}})} |x(1-x)^\nu|\; \|e^0\|\,.$$

We consider in the following the eigenvalue problem $G^{-1} A_{\mathrm{sym}} x = \lambda x$, $x^T := (x_S^T, x_N^T)$, and prove $0 \le \lambda \le 1$. To start, we give the block structure of $G^{-1} A_{\mathrm{sym}}$

$$\begin{aligned} G^{-1} A_{\mathrm{sym}} &= \begin{pmatrix} G_S^{-1} & -G_S^{-1} B_{SN} D^{-1} \\ -D^{-1} B_{NS} G_S^{-1} & \frac{3}{2} D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} + \alpha \mathrm{Id} \end{pmatrix} \begin{pmatrix} A_{SS} & A_{SN} \\ A_{NS} & A_{NN} \end{pmatrix} \\ &= \begin{pmatrix} G_S^{-1} A_{SS} - G_S^{-1} B_{SN} D^{-1} A_{NN} D^{-1} B_{NS} & 0 \\ \begin{array}{c} \frac{3}{2} D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} A_{NN} D^{-1} B_{NS} \\ + \alpha A_{NS} - D^{-1} B_{NS} G_S^{-1} A_{SS} \end{array} & \begin{array}{c} \frac{1}{2} D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} A_{NN} \\ + \alpha A_{NN} \end{array} \end{pmatrix} \end{aligned}$$

Here, we have used $A_{SN} = B_{SN} D^{-1} A_{NN}$. Obviously, we find $\lambda > 0$ since G^{-1} and A_{sym} are symmetric and positive definite. Let us first assume $x_S \ne 0$. Then, λ is an eigenvalue of $G^{-1} A_{\mathrm{sym}}$ for the eigenvector x only if x_S is an eigenvector of the lower dimensional problem

$$(A_{SS} - B_{SN} D^{-1} A_{NN} D^{-1} B_{NS}) x_S = \lambda G_S x_S\,.$$

Here, we have used the fact that $G^{-1}A_{\text{sym}}$ is a lower block triangular matrix. By means of $x_S^T G_S x_S \geq x_S^T A_{SS} x_S$ and $A_{NN} > 0$, we get

$$x_S^T G_S x_S \geq x_S^T A_{SS} x_S - x_S^T B_{SN} D^{-1} A_{NN} D^{-1} B_{NS} x_S = \lambda x_S^T G_S x_S \,,$$

and thus $\lambda \leq 1$. Let us now consider the second case $x^T = (0, x_N^T)$, i.e. $x_S = 0$. Then, the eigenvalue problem $G^{-1}A_{\text{sym}}x = \lambda x$ can be reduced to the following eigenvalue problem on a smaller space

$$(\frac{1}{2} D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} + \alpha \text{Id}) x_N \;=\; \lambda A_{NN}^{-1} x_N \,.$$

Observing $w^T A_{\text{sym}} w > 0$, we find for $w = (A_{SS}^{-1} B_{SN} D^{-1} x_N, -A_{NN}^{-1} x_N)$

$$x_N^T D^{-1} B_{NS} A_{SS}^{-1} B_{SN} D^{-1} x_N \;\leq\; x_N^T A_{NN}^{-1} x_N \,.$$

Using the assumption (4.4) on G_S and the definition of α, a straightforward calculation shows

$$\begin{aligned}
\lambda x_N^T A_{NN}^{-1} x_N &= \frac{1}{2} x_N^T D^{-1} B_{NS} G_S^{-1} B_{SN} D^{-1} x_N + \alpha x_N^T x_N \\
&\leq \frac{1}{2} x_N^T D^{-1} B_{NS} A_{SS}^{-1} B_{SN} D^{-1} x_N + \frac{1}{2} x_N^T A_{NN}^{-1} x_N \\
&\leq x_N^T A_{NN}^{-1} x_N \,.
\end{aligned}$$

Since A_{NN}^{-1} is symmetric positive definite, we find $\lambda \leq 1$. By means of the positive definiteness of G and A_{sym}, we finally obtain $0 \leq \lambda \leq 1$ and by means of Lemma 4.4

$$\|A_{\text{sym}} e^\nu\| \;\leq\; \frac{\|G\|}{e\nu} \|e^0\| \;\leq\; \frac{C}{\nu} \|e^0\| \,.$$

□

In contrast to [BD98, BDL99, BDW98], no Schur complement system has to be solved exactly to guarantee $B^T y^\nu = 0$ in each smoothing step. The action of G^{-1} can be easily obtained by applying G_S^{-1}.

REMARK 4.6. *The second column of G^{-1} is redundant for the numerical realization of the smoother and can be replaced by zero. Furthermore in Section 5, we will show that G in (4.6) can be replaced by a Gauß–Seidel smoother.*

In addition to the smoothing and approximation property, the $\|\cdot\|$–stability of the iteration matrix $\text{Id} - G^{-1}A_{\text{sym}}$ is required for the convergence of the $\mathcal{W}$–cycle, see e.g. [BS94, Hac85]. As in case of the smoothing property, the stability is inherited from G_S^{-1}.

LEMMA 4.7. *Under the assumption 4.4, the iteration (4.6) is stable and the iteration error e^ν is bounded by*

$$\|e^\nu\| \;\leq\; C\,\|e^0\|$$

with a constant independent of ν.

Proof. Having in mind that the iteration error e^ν can be written as

$$e^\nu = (\text{Id} - G^{-1} A_{\text{sym}})^\nu e^0 \,,$$

$\|e^\nu\|$ can be bounded by

$$\|e^\nu\| \;\leq\; \|G^{-1/2}\| \, \|\text{Id} - G^{-1/2} A_{\text{sym}} G^{-1/2}\,\|^\nu \, \|G^{1/2}\| \, \|e^0\| \,.$$

By means of $\sigma(G^{-1}A_{\text{sym}}) \subset [0,1]$, we find $\|e^\nu\| \leq \sqrt{\kappa(G)}\,\|e^0\|$. Then, the stability of the iteration (4.6) follows immediately from Lemma 4.4. □

Now, we are in the setting to formulate our main result. Our multigrid method will be defined in terms of the smoothing iteration (4.6), the coarse grid correction w_{l-1} defined by (4.2) and the standard prolongation I_{l-1}^l for the coarse grid correction. The start iterate y_l^0 on each level for the defect equation is set to zero as usual guaranteeing $B_l^T y_l^0 = 0$.

THEOREM 4.8. *Under the assumption (4.4), the convergence rate for the $\mathcal{W}$–cycle is independent of the number of refinement levels provided that the number of smoothing steps is large enough.*

Proof. The proof follows from the approximation property, Lemma 4.3, the smoothing property, Lemma 4.5 and the stability, Lemma 4.7. For details, we refer for example to [BS94, Hac85]. □

5. Numerical results. The multigrid algorithm on the unconstrained product space has been implemented for two space dimensions in the framework of the finite element toolbox UG [BBJ+97]. We present some numerical experiments illustrating the performance of the algorithm. The theoretical results for the $\mathcal{W}$–cycle are confirmed by the numerical ones. Furthermore, we obtain level independent convergence rates for the $\mathcal{V}$–cycle in case of full regularity. We use piecewise linear Lagrangian finite elements and the dual space M_h^2 for the Lagrange multiplier introduced in Subsection 2.1.

Three different examples are considered: The first one with full regularity on the unit square decomposed into nine subdomains having four crosspoints. The second example is characterized by a solution with a corner singularity, and the third example involves highly discontinuous coefficients. We refer to [Woh98, Woh99a] for the numerical comparison of the discretization errors for different Lagrange multiplier spaces in case of these examples.

Let us consider the examples in more detail. Example 1 is given by $-\Delta u = f$ on $(0,1)^2$, where the right hand side f and the Dirichlet boundary conditions are chosen so that the exact solution is $(\exp(-500s)-1)*(\exp(-500t)-1)*(\exp(-500q)-1)*(1-3r)^2$. Here $s := (x-1/3)^2$, $t := (x-2/3)^2$, $q := (y-1/2)^2$ and $r := (x-1/2)^2+q$. The solution and the initial triangulation are shown in Figure 5.1.

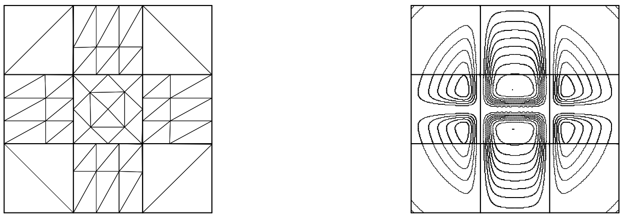


FIG. 5.1. *Decomposition and initial triangulation (left) and solution (right) (Example 1)*

The domain is decomposed into nine subdomains defined by $\Omega_{ij} := ((i-1)/3, i/3) \times((j-1)/3, j/3)$, $1 \leq i,j \leq 3$ and the triangulations do not match at the interfaces. We observe two different situations at the interface, e.g. the isolines of the solution are almost parallel at $\partial\Omega_{11} \cap \partial\Omega_{12}$ whereas at $\partial\Omega_{11} \cap \partial\Omega_{21}$ the angle between the isolines and the interface is bounded away from zero on a large part of the interface.

In case that the isolines are orthogonal on the interface the exact Lagrange multiplier will be zero.

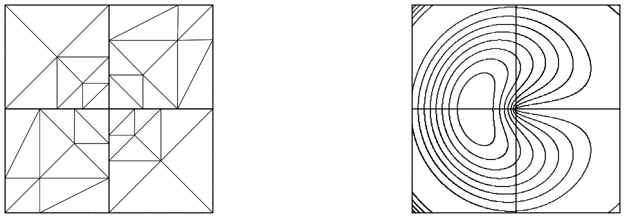


FIG. 5.2. *Decomposition and initial triangulation (left) and solution (right) (Example 2)*

In our second example, we consider the union square with a slit at $y = 0.5$, $0.5 \leq x \leq 1$, decomposed into four subdomains, see Figure 5.2. Here, the right hand side f and the Dirichlet boundary conditions of $-\Delta u = f$ are chosen so that the exact solution is given by $(1 - 3r^2)^2 r^{1/2} \sin(1/2\phi)$, where $x - 1/2 = r\cos\phi$, and $y - 1/2 = r\sin\phi$. The solution has a singularity in the center of the domain. We do not have H^2–regularity, and we therefore cannot expect an $\mathcal{O}(h)$ behaviour for the discretization error in the energy norm.

Our last example illustrates the influence of discontinuous coefficients. We consider the diffusion equation $-\operatorname{div} a\nabla u = f$, on $(0,1)^2$, where the coefficient a is discontinuous. The unit square Ω is decomposed into four subdomains $\Omega_{ij} := ((i-1)/2, i/2) \times ((j-1)/2, j/2)$ as in Figure 5.3.

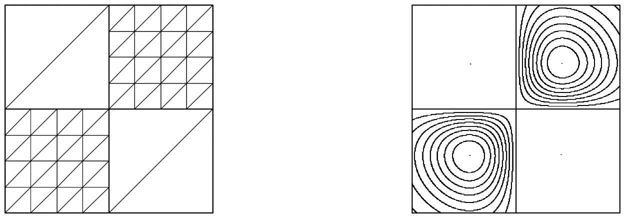


FIG. 5.3. *Decomposition and initial triangulation (left) and solution (right) (Example 3)*

The coefficients on the subdomains are given by $a_{11} = a_{22} = 0.00025$, $a_{12} = a_{21} = 1$. The right hand side f and the Dirichlet boundary conditions are chosen to match a given exact solution, $(x - 0.5)(y - 0.5)\exp(-10((x - 0.5)^2 + (y - 0.5)^2))/a$. In particular, f does not reflect the discontinuity of the coefficient a. This solution is continuous with vanishing $[a\nabla u\mathbf{n}]$ on the interfaces. Because of the discontinuity of the coefficients, we use a highly non–matching triangulation at the interface, see Figure 5.3.

We use a standard uniform refinement technique for simplicial triangulations on the different subdomains. Each element is decomposed into four congruent subtriangles. The numerical results are based on the multigrid algorithm introduced in Section 4. In particular, we use the level dependent transfer operator given by (4.3). We compare different smoothing operators and the influence of the number of smoothing steps on the convergence rates. We discuss a symmetric Gauß–Seidel smoother for G_S as well as a non–symmetric one.

The start iterate y_l^0 for the defect iteration on each level is set to zero, and the

ν–th smoothing step on level l is given by

$$y_l^\nu := y_l^{\nu-1} + \begin{pmatrix} G_S & 0 \\ D^{-1}B_{NS} & \mathrm{Id} \end{pmatrix}^{-1} \left(d_l - A_{\mathrm{num}} y_l^{\nu-1}\right), \tag{5.1}$$

where A_{num} is defined in (3.11). For simplicity, we suppress the level index in case of the matrices if the level l is clear from the context. A straightforward calculation as in the proof of Lemma 4.5 shows that $B_l^T y_l^\nu = 0$, and that (5.1) can be replaced by (4.6). Thus if G_S satisfies the assumptions (4.4), we can apply Lemma 4.5. We remark that the action of the non–symmetric smoother in (5.1) is due to the special structure of the defect d_l the same as of the symmetric one given by (4.5), and we are in the setting of Subsection 4.3.

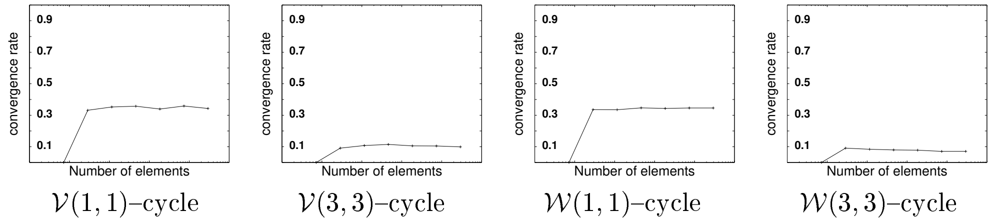


FIG. 5.4. *Asymptotic convergence rates for the symmetric Gauß–Seidel smoother (Example 1)*

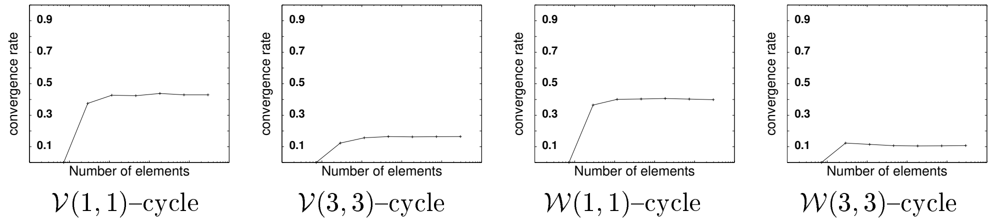
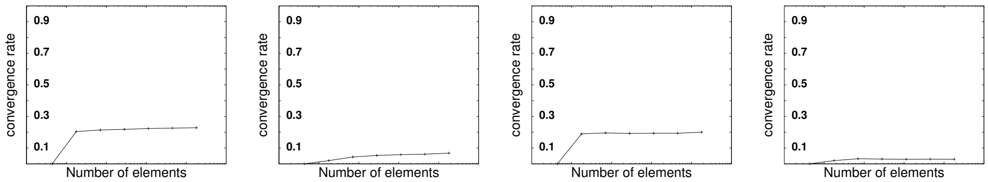


FIG. 5.5. *Asymptotic conv. rates for the non–symmetric Gauß–Seidel smoother (Example 1)*

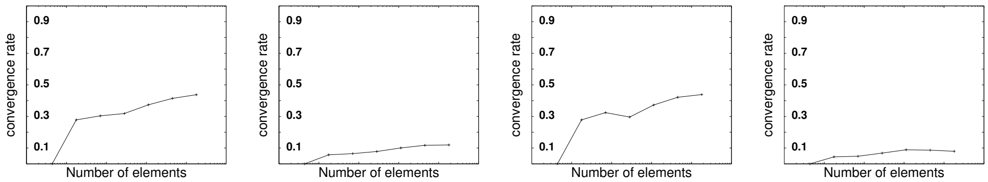
We consider two different choices of G_S. In the first case, G_S is a symmetric Gauß–Seidel smoother where the unknowns u_S are ordered lexicographically. Then, Theorem 4.8 yields a level independent convergence rate for the $\mathcal{W}$–cycle provided that the number of smoothing steps is large enough. In the second case, we replace the symmetric Gauß–Seidel smoother G_S in the unknowns u_S by a non–symmetric Gauß–Seidel, again the unknowns are ordered lexicographically. Then, the iteration step (5.1) is nothing else than a standard non–symmetric Gauß–Seidel iteration where the unknowns are ordered as follows (u_S, u_N). Thus, no additional work is necessary to obtain iterates in the constrained space. We remark that the ordering is important. Only if the unknowns u_N are considered after u_M, the constraints at the interfaces are satisfied in each smoothing step. The only difference compared with a standard multigrid algorithm for symmetric positive definite problems on nested spaces is the transfer operator. In our case, we replace the standard grid transfer operator I_l^{l-1} by the level dependent one $(I_{\mathrm{mod}})_l^{l-1} = (\mathrm{Id} - B_{l-1} W_{l-1}^T)\, I_l^{l-1}$ defined in (4.3). Observing that $B_{l-1} W_{l-1}^T$ is not only sparse but the number of non–zero entries is bounded by $C\sqrt{n}$ in 2D and by $Cn^{2/3}$ in 3D, we find that the extra amount of work is of smaller order than one smoothing step. Here, n is the number of unknowns. In particular, the application of $(I_{\mathrm{mod}})_l^{l-1}$ involves only a scaled mass–matrix on the interfaces.

Figures 5.4–5.9 show the convergence rates of the $\mathcal{V}$–cycle and the $\mathcal{W}$–cycle for the two different smoothers in dependency of the number of elements. In 5.4, 5.6 and 5.8, we use the symmetric Gauß–Seidel smoother whereas in 5.5, 5.7 and 5.9, the non–symmetric one is applied. Both smoothers can be easily applied and no extra work is required to guarantee the constraints at the interface. The two pictures from the left in the figures show the results for a $\mathcal{V}$–cycle whereas the two pictures from the right show the corresponding results for the $\mathcal{W}$–cycle. The number of pre– and post–smoothing steps is equal ν. Here, we consider the cases $\nu = 1$ and $\nu = 3$.



$\mathcal{V}(1,1)$–cycle $\mathcal{V}(3,3)$–cycle $\mathcal{W}(1,1)$–cycle $\mathcal{W}(3,3)$–cycle

FIG. 5.6. *Asymptotic convergence rates for the symmetric Gauß–Seidel smoother (Example 2)*

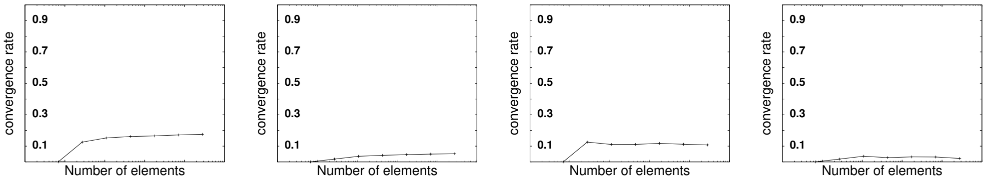


$\mathcal{V}(1,1)$–cycle $\mathcal{V}(3,3)$–cycle $\mathcal{W}(1,1)$–cycle $\mathcal{W}(3,3)$–cycle

FIG. 5.7. *Asymptotic conv. rates for the non–symmetric Gauß–Seidel smoother (Example 2)*

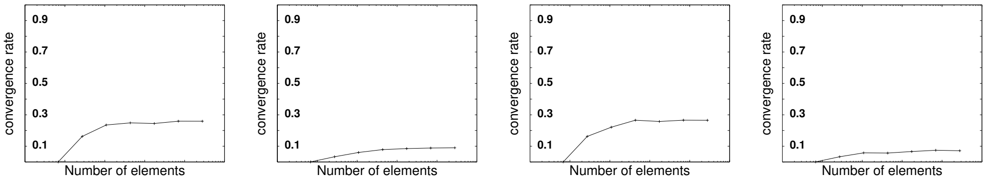
We start with a comparison of the numerical results for Examples 1 and 3, see Figures 5.4, 5.5, 5.8 and 5.9. Here, we have full regularity, and obtain level independents convergence rate for all test settings, even in case of the $\mathcal{V}$–cycle with one pre– and one post–smoothing step and for the non–symmetric Gauß–Seidel smoother. Increasing the number of smoothing steps from one to three improves the convergence rate for the $\mathcal{W}$–cycle and $\mathcal{V}$–cycle considerably. The numerical results show that the convergence rates are approximately by the factor three smaller in case of three smoothing steps. The convergence rates for the $\mathcal{W}$–cycles in these two examples are only minimal better than the ones for the $\mathcal{V}$–cycles. The results for the symmetric smoother are better than for the non–symmetric one. However, one application of the symmetric variant of G_S^{-1} is as twice as expensive as one of the non–symmetric one. The effect of the choice of the smoother is for Example 3 more distinct than for Example 1. Finally, we consider the results for Example 2 in more detail. Figures 5.6 and 5.7 show the corresponding convergence rates. Here, we have no H^2–regularity. For the case $\nu = 3$, we obtain level independent convergence rates for the symmetric Gauß–Seidel smoother as well as for the unsymmetric one, see Figure 5.6. In case of the unsymmetric smoother and $\nu = 1$, it seems to be that the convergence rates get asymptotically constant. But the asymptotic starts late. Increasing the number of smoothing steps gives not only quantitative better results but also the asymptotic starts earlier, see Figure 5.7.

Finally, we remark that the defect restriction $d_{l-1} = (I_{\text{mod}})_l^{l-1}\, d_l$ is only one



$\mathcal{V}(1,1)$–cycle $\mathcal{V}(3,3)$–cycle $\mathcal{W}(1,1)$–cycle $\mathcal{W}(3,3)$–cycle

FIG. 5.8. *Asymptotic convergence rates for the symmetric Gauß–Seidel smoother (Example 3)*



$\mathcal{V}(1,1)$–cycle $\mathcal{V}(3,3)$–cycle $\mathcal{W}(1,1)$–cycle $\mathcal{W}(3,3)$–cycle

FIG. 5.9. *Asymptotic conv. rates for the non–symmetric Gauß–Seidel smoother (Example 3)*

possibility. A more general form of the defect on level $l-1$ can be obtained by

$$d_{l-1} := (\mathrm{Id} - B_{l-1} W_{l-1}^T)\, I_l^{l-1}\, (d_l - B_l z_l)$$

where z_l is a suitable vector being in $\mathbb{R}^m$ and m is the dimension of the Lagrange multiplier space on level l. To find appropriate vectors z_l, we have to consider the saddle point formulation. We recall that the approximation property 4.3 is obtained by a priori estimates for the corresponding saddle point formulation. The local elimination of the Lagrange multiplier was the starting point for the introduction of our new setting. The choice $\lambda_l = W_l^T\,(f_l - Au_l)$ yields the defect $(d_l, 0)^T$ for the saddle point problem, where d_l is the defect of the corresponding system on the unconstrained product space. Observing that λ_l is an approximation for the flux, other choices not depending on f_l might be reasonable. Furthermore, the components of $W^T f_l$ are of the order h_l and tend to zero with increasing refinement level l. Setting $\lambda_l := -(0, -\alpha_1 D_l^{-2} M_l, \alpha_2 D_l^{-1}) A_l\, u_l$ with non–negative weights α_1 and α_2, $\alpha_1 + \alpha_2 = 1$, gives $z_l = -W_l^T f_l + \alpha_1 D_l^{-2} B_l^T A_l\, u_l$.

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