Devaney chaos by iterated differentiation of analytical functions

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Abstract

By checking the relevant conditions we prove that the common operator of differentiation, acting in a suitable metricized space of analytic functions, is chaotic in the sense of Devaney even though it is a linear operator. To invalidate the possible objection that in view of this operator's discontinuity there is nothing to wonder about we show by an example that discontinuity of a mapping and sensitivity are independent.

Keywords: chaos, differential operator, hyperbolic functions, butterfly-effect
MSC 58 Global Analysis, MSC 47 Operator Theory

1 Introduction

By E we denote the complex unit circle $E := \{x \in \mathbb{C} | |x| \leq 1\}$. Then the vector space $C(E,\mathbb{C})$ of the functions $f : E \to \mathbb{C}$ which are continuous in E, together with the maximum-norm

$$\|f\| := \max_{x \in E} |f(x)|, \tag{1.1}$$

is a normed vector space. Now we choose a suitable subset $A \subset C(E,\mathbb{C})$:
We consider analytic functions f which admit a representation by a power-series with coefficients a_i in $\mathbb{C}$,

$$f(z) = \sum_{i=0}^{\infty} a_i \frac{(z - z_0)^i}{i!},$$

around a point $z_0 \in \mathbb{C}$ with a radius of convergence $R_f > r > 0$. Without loss of generality we present the theory for $z_0 := 0$ and $r := 1$. With $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we have

$$A := \left\{ f : E \to \mathbb{C} | \ f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}, \text{ the series convergent for } |x| < R_f, R_f > 1 \right\}. \quad (1.2)$$

Each $f \in A$ is infinitely often differentiable in E and its Taylor series is absolutely convergent for $x \in E$. The set A is a normed vector space by (1.1) and a subspace of $C(E,\mathbb{C})$.
By the differential operater $D : A \to A$, which assigns to each function $f \in A$ its first derivative $f' \in A$, we construct a dynamical system:

$$\begin{aligned} D^0(f) &:= f \in A, \ f' = D^1(f) = D(f), f'' = D^2(f) = D(f'), \ldots, \\ D^n(f) &= D(D^{n-1}(f)) = f^{(n)}, \end{aligned}$$

where D^n means the n-fold composition of the differential operator D. Hence for $f \in A$, and $x \in E$ we have:

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} a_i \tfrac{x^i}{i!} &&= a_0 + a_1 x + a_2 \tfrac{x^2}{2!} + a_3 \tfrac{x^3}{3!} + \cdots \\ D(f)(x) = f'(x) &= \sum_{i=0}^{\infty} a_{i+1} \tfrac{x^i}{i!} &&= a_1 + a_2 x + a_3 \tfrac{x^2}{2!} + a_4 \tfrac{x^3}{3!} + \cdots \end{aligned}$$

and for the n-th derivative $D^n(f) = f^{(n)}$ we have

$$D^n(f)(x) = \sum_{i=0}^{\infty} a_{i+n} \frac{x^i}{i!} = a_n + a_{n+1} x + a_{n+2} \frac{x^2}{2!} + a_{n+3} \frac{x^3}{3!} + \cdots \quad (1.3)$$

Thus, applying the differential operator D to the Taylor series of f has the same effect as applying the shift operator S defined by

$$S((a_0, a_1, a_2, a_3, \ldots)) := (a_1, a_2, a_3, \ldots) \quad (1.4)$$

to the 'Taylor sequence' $(a_i | i \in \mathbb{N}_0)$ of Taylor coefficients of f.
Denoting by e_α for $\alpha \in \mathbb{C}$ the exponential function

$$e_\alpha : E \to \mathbb{C}, \quad x \mapsto \alpha e^x, \quad (1.5)$$

we have with e_α a fixed element of the differential operator D.

2 Statements and Results

First we note a well-known theorem, which essentially says that in differential calculus there occurs the butterfly effect.

Theorem 1

The differential operator D is discontinuous. (2.1)

Let $f \in A, \varepsilon > 0, K > \varepsilon\pi, a := \frac{\varepsilon}{2\cosh(2K/\varepsilon)}$.

$$g : E \to \mathbb{C} \quad \textit{with} \quad g(x) := f(x) - a\ \sin(2Kx/\varepsilon). \tag{2.2}$$

Then $g \in A$ *and*

$$\|f - g\| < \varepsilon \quad \textit{and} \quad \|D(f) - D(g)\| = K. \tag{2.3}$$

In Figure 1 for real values $x \in [-1, 1]$ f and g are graphically represented.
The differential operator knows

fixed elements:	e_α, see (1.5),
periodic elements of period 2:	$\sinh, \cosh, x \mapsto e^{-x}$,
and of period 4:	$\sin, \cos$.

Remembering the Feigenbaum-diagramm in Peitgen et al. 1994, p.133, we ask for period 8, 16, 32,... and for period 3.
By Π_n and Π we denote for $n \in \mathbb{N}$ the solution of the differential equation $y = y^{(n)}$:

Definition 1

$$\Pi_n := \{f \in A | D^n(f) = f\}, \quad \Pi := \bigcup_{n=1}^{\infty} \Pi_n. \tag{2.4}$$

The sets Π_n and Π are both linear subspaces of A, their elements are periodic elements of the differential operator D. We shall show that Π_n contains more elements than the exponential functions (1.5).
In 1955 A. Erdelyi et al. described, under the headline "Miscellaneous Functions" functions, which they called "hyperbolic functions of higher order" because of their similarity with the hyperbolic functions sinh and cosh. To our knowledge, they had fallen in oblivion, now they will have a come-back:

Theorem 2

For the differential operator D there exist elements of period 3. (2.5)

For the differential operator D there exist elements of period n for each $n \in \mathbb{N}$ (2.6)

The functions $h_{n,k} : E \to \mathbb{C}$, $n \in \mathbb{N}$, $k = 0, 1, \ldots, n-1$, defined by

$$h_{n,0}(x) := \sum_{i=0}^{\infty} \frac{x^{in}}{(in)!} = 1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \cdots \tag{2.7}$$

and

$$h_{n,k} := D^k(h_{n,0}) \tag{2.8}$$

fulfil

$$h_{n,k} \in \Pi_n. \tag{2.9}$$

In Figure 2 for real arguments the functions $h_{1,0}$, $h_{2,0}$, $h_{3,0}$ and $h_{4,0}$ are graphically displayed.
Remembering Li & Yorke's paper "period 3 implies chaos" of 1975 we cannot conclude chaos of the differential operator D because the authors made use of the continuity of their iteration function φ. This conclusion is prevented by Theorem 1 (2.1). For the same reason Theorem 3.11 of Sharkovsky 1993 fails.
From (1.3) we directly conclude that the elements of Π_n are functions with periodic Taylor sequence:

Lemma 1 *Let* $f \in A$, $f(x) = \sum\limits_{i=0}^{\infty} a_i \frac{x^i}{i!}$. *Then*

$$f \in \Pi_n \iff a_{n+i} = a_i \text{ for } i \in \mathbb{N}_0. \tag{2.10}$$

There is another criterion for chaos, the existence of homocline orbits, see Figure 3:

Theorem 3
Each fixed element e_α, *see* (1.5), *of the differential operator* D *is homoclinic, the sequence of functions* $(q_i \in A | i \in \mathbb{Z})$ *with*

$$q_i(x) := \begin{cases} \alpha e^x & \text{if } i \in \mathbb{N} \\ \alpha e^x + \frac{x^{-i}}{(-i)!} & \text{if } i \in \mathbb{Z} \setminus \mathbb{N} \end{cases} \tag{2.11}$$

forms a non-degenerate homoclinic orbit.

Marotto 1978 proved that the existence of a homoclinic orbit implies chaos in the sense of Li & Yorke. In his proof he used the continuity of the iteration function, but our iteration operator D is discontinuous, see (2.1). Therefore Theorem 3 fails to lead to chaos. So we have to use a basic definition. In 1989 Robert Devaney characterized a chaotic function $\varphi : M \to M$ by means of three criteria:

φ is *sensitive* and
topologically transitive and
its periodic points are dense in M.

Formulated for a normed vector space X his definitions read:

Definition 2 *(Butterfly-effect)*
Let $M \subset X$. The map $\varphi : M \to M$ is said to be *sensitive* if there exists $\delta > 0$ such that, for any $x \in M$ and any neighborhood $U(x)$ of x, there exists $y \in U(x)$ and $n \in \mathbb{N}$ such that

$$\|\varphi^n(x) - \varphi^n(y)\| > \delta. \tag{2.12}$$

Definition 3 *(Mixture-property)* See Figure 3.
Let $M \subset X$. Then $\varphi : M \to M$ is said to be *topologically transitive* if for any pair of open sets $U \subset M, V \subset M$ there exists $n \in \mathbb{N}$ such that

$$\varphi^n(U) \cap V \neq \emptyset. \tag{2.13}$$

For X we chose $C(E,\mathbb{C})$ with the norm (1.1), for M we take A (1.2) and for the map φ the differential operator $D : A \to A$:

Theorem 4
The differential operator $D : A \to A$ is chaotic in the sense of Devaney, it is

sensitive, (2.14)

topologically transitive and (2.15)

its periodic points Π (2.4) are dense in A. (2.16)

So the dynamical system generated by the differential operator is chaotic.

Example:
At last we demonstrate the independence of discontinuity and sensitivity, see Definition 2:
We define an iteration function $f : [0,1] \to [0,1]$ of the Dirichlet-type, by

$$a := \frac{1}{2} - \frac{1}{6}\sqrt{3} = 0.2113\ldots \quad \text{and} \qquad b := \frac{1}{2} + \frac{1}{6}\sqrt{3} = 0.7886\ldots$$

we define

$$f(x) := \begin{cases} 4(x-\frac{1}{2})^3 + \frac{1}{2} & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,a] \setminus \mathbb{Q} \\ \frac{1}{2} & \text{if } x \in (a,b) \setminus \mathbb{Q} \\ 1 & \text{if } x \in [b,1] \setminus \mathbb{Q}, \end{cases} \tag{2.17}$$

see figure 5. This function is continuous at its fixed points 0, $\frac{1}{2}$ and 1, otherwise discontinuous.
In the intervalls $[0,a]$ and $[b,1]$ the function f is sensitive: an orbit with starting point $x \in [0,a] \cap \mathbb{Q}$ runs to the fixed point $\frac{1}{2}$, because all of its elements are rational numbers. But an orbit with starting point $y \in [0,a] \setminus \mathbb{Q}$ is eventually periodic, it ends in the fixed point 0. In intervall $[b,1]$ we find an analoguous behaviour.
An orbit with starting point $x \in (a,b) \cap \mathbb{Q}$ converges to the fixpoint $\frac{1}{2}$, meeting there orbits with starting point $y \in (a,b) \setminus \mathbb{Q}$.
Choose $x_0 \in (a,b)$ and an ε-neighborhood U of x_0 containing the fixed point $\frac{1}{2}$. In every case the orbit $(x_n)_{n \in \mathbb{N}_0} = \left(f^n(x_0)\right)_{n \in \mathbb{N}}$ is subset of U, it does not walk away from the fixed point $\frac{1}{2}$. There is no $\delta > 0$ with property (2.12), therefore function f is not sensitive, compare Definition 2.

3 Proofs of Theorems 1,2,3:

Proof of Theorem 1 With (2.2) and (1.1) we have

$$\begin{aligned}
\|f-g\| &= a \max_{x\in E} |\sin(2Kx/\varepsilon)| \\
&= a \max_{x\in E} \sqrt{\sin^2(Re(2Kx/\varepsilon)) + \sinh^2(Im(2Kx/\varepsilon))} \\
&\leq a \sqrt{1+\sinh^2(2K/\varepsilon)} = a \cosh(2K/\varepsilon) \\
&= \tfrac{\varepsilon}{2} < \varepsilon, \\
\|D(f)-D(g)\| &= a \max_{x\in E} |\tfrac{2K}{\varepsilon}\cos(2Kx/\varepsilon)| = \tfrac{2Ka}{\varepsilon} \max_{x\in E} |\cos(2Kx/\varepsilon)| \\
&= \tfrac{2Ka}{\varepsilon} \max_{x\in E} \sqrt{\cos^2(Re(2Kx/\varepsilon)) + \sinh^2(Im(2Kx/\varepsilon))} \\
&= \tfrac{2Ka}{\varepsilon}\sqrt{1+\sinh^2(2K/\varepsilon)} = \tfrac{2Ka}{\varepsilon}\cosh(2K/\varepsilon) = \\
&= K.
\end{aligned}$$

Thus we have proved (2.3). Because of $K > \varepsilon\pi$ we have (2.1).
(2.3) shows that functions f and g with small distance ε may have derivatives $D(f)$ and $D(g)$ which have arbitrarily great distance K. So we have the butterfly effect.

qed.

Proof of Theorem 2:
The summands of the series (2.7) form a subsequence of the summands of the exponential series, hence $h_{n,0} \in A$ and $h_{n,k} \in A$.
Termwise derivation of (2.7) shows that, for each $n \in I\!N$, we have

$$D^n(h_{n,0}) = h_{n,n} = h_{n,0},$$

i.e., $h_{n,0} \in \Pi_n$, see (2.4), and therefore for the derivatives $h_{n,k}$, see (2.8), we have

$$h_{n,k} \in \Pi_n \quad \text{for each} \quad k = 0, 1, \ldots, n-1.$$

Thus we have (2.9), and (2.6) and (2.5).

qed.

Proof of Theorem 3:
We check the properties of Devaney 1989, 16.6. The sequence $(q_i | i \in \mathbb{Z})$, see (2.11), satisfies

1. $q_{i+1} = D(q_i)$:

For $i \in \mathbb{N}_0$ this is trivial, so let $i \in \mathbb{Z} \setminus \mathbb{N}_0$. We use the linearity of the differential operator:

$$\begin{aligned} D(q_i)(x) &= D\left(\alpha e^x + \tfrac{x^{-i}}{(-i)!}\right) = D(\alpha e^x) + D\left(\tfrac{x^{-i}}{(-i)!}\right) = \alpha e^x + \left(\tfrac{x^{-i-1}}{(-i-1)!}\right) \\ &= q_{i+1}(x). \end{aligned}$$

The sequence $(q_i | i \in \mathbb{Z})$ satisfies

2. $q_i = e_\alpha$ for $i \in \mathbb{N}$ and

3. $\lim q_i = e_\alpha$ for $i \to -\infty$,

since the additional term $\frac{x^{-i}}{(-i)!}$ tends to zero. Thus e_α is homoclinic and $(q_i | i \in \mathbb{Z})$ is homoclinic orbit.

4. Theorem 1 guarantees the non-degeneracy, because there is no neighborhood of any q_i where the differential operator acts as a constant mapping. So we have proved Theorem 3.

4 Proof of Theorem 4:

(2.14) follows from Theorem 1 (2.3) if we choose $n := 1$ and $\delta \in (\varepsilon\pi, K)$. Thus the differential operator D is sensitive, see (2.12).
The topological transitivity (2.13) means, relative to the differential operator, that the set of the n-th derivatives $f^{(n)}(U)$ of an arbitrarily small open subset $U \subset A$ are spread over the whole space A, for increasing n, see Figure 3. We prove the topological transitivity for D in a sharper way:

Proposition 1 *Let $f \in A, g \in A$ and let U_ε be an ε-neighborhood of f in the norm-topology. Then there exists a function $h \in U_\varepsilon$ and $n \in \mathbb{N}$ with*

$$D^n(h) = g. \tag{4.1}$$

Proof of (4.1): See Figure 4.
The functions f and g have Taylor series around 0 with radius of convergence $R_f > 1, R_g > 1$:

$$f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}, \qquad g(x) = \sum_{i=0}^{\infty} b_i \frac{x^i}{i!}$$

with $a_i, b_i \in \mathbb{C}$. By $(R_n \in A | n \in \mathbb{N})$ we denote the sequence of remainders of partial sums of f:

$$R_n(x) := \sum_{i=n}^{\infty} a_i \frac{x^i}{i!} = f(x) - \sum_{i=0}^{n-1} a_i \frac{x^i}{i!}. \tag{4.2}$$

Because the series of f for $x \in E$ is absolutely convergent $(\|R_n\|/n \in \mathbb{N})$ is zero-sequence. Therefore we can find $k \in \mathbb{N}$ with

$$\|R_n\| < \varepsilon/2 \quad \text{for} \quad n \geq k. \tag{4.3}$$

Now we take the Taylor sequence $(b_i \in \mathbb{C} | i \in \mathbb{N}_0)$ of g and define for $n \in \mathbb{N}$ the function $S_n : E \to \mathbb{C}$ by

$$\begin{aligned} S_n(x) &:= \sum_{i=0}^{\infty} b_i \frac{x^{n+i}}{(n+i)!} = b_0 \frac{x^n}{n!} + b_1 \frac{x^{n+1}}{(n+1)!} + b_2 \frac{x^{n+2}}{(n+2)!} + \cdots \\ &= \frac{x^n}{n!} \left[b_0 + \sum_{i=1}^{\infty} b_i \frac{x^i}{(n+1)(n+2)\dots(n+i)} \right]. \end{aligned} \tag{4.4}$$

For the n-th derivation $D^n(S_n)$ of S_n we find by termwise derivation

$$D^n(S_n)(x) = D^n \left(\sum_{i=0}^{\infty} b_i \frac{x^{n+i}}{(n+i)!} \right) = \sum_{i=0}^{\infty} b_i \frac{x^i}{i!} = g(x).$$

Hence, the series for S_n also has radius of convergence $R_g > 1$ and $S_n \in A$. Looking for an upper bound of (4.4) we find

$$\begin{aligned} \|S_n\| = \max_{x \in E} |S_n(x)| &\leq \max_{x \in E} \frac{|x|^n}{n!} \left[|b_0| + \sum_{i=1}^{\infty} |b_i| \frac{|x|^i}{(n+1)(n+2)\dots(n+i)} \right] \\ &\leq \max_{x \in E} \frac{|x|^n}{n!} \left[|b_0| + \sum_{i=1}^{\infty} |b_i| \frac{|x|^i}{i!} \right] \\ &\leq \frac{1}{n!} \sum_{i=0}^{\infty} \frac{|b_i|}{i!}. \end{aligned}$$

The last sum has a finite value, because the series for g is absolutely convergent if $|x| \leq 1$. With

$$K := \sum_{i=0}^{\infty} \frac{|b_i|}{i!}$$

we have for $n \in I\!N$:

$$\|S_n\| \leq \frac{K}{n!}.$$

$\left(\frac{K}{n!} | n \in I\!N\right)$ being a zero-sequence, we can find $m \in I\!N$ with

$$\|S_n\| \leq \frac{K}{n!} < \varepsilon/2 \quad \text{for} \quad n \geq m. \tag{4.5}$$

Now we put $n := \max\{k, m\}$ with k from (4.3) and m from (4.5), and with the Taylor sequences of f and g we define a new sequence:

$$\begin{aligned} c_i &:= a_i \quad \text{for} \quad i = 0, 1, \ldots, n-1 \quad \text{and} \\ c_i &:= b_{i-n} \quad \text{for} \quad i \geq n, \end{aligned} \tag{4.6}$$

i.e.,

$$(c_i \in \mathbb{C} | i \in I\!N_0) = (a_0, a_1, a_2, \ldots, a_{n-1}, b_0, b_1, b_2, b_3, \ldots).$$

This sequence defines a Taylor series and a function $h : E \to \mathbb{C}$ by

$$\begin{aligned} h(x) := \sum_{i=0}^{\infty} c_i \tfrac{x^i}{i!} &= \sum_{i=0}^{n-1} a_i \tfrac{x^i}{i!} + \sum_{i=n}^{\infty} b_{i-n} \tfrac{x^i}{i!} \\ &= \sum_{i=0}^{n-1} a_i \tfrac{x^i}{i!} + \sum_{i=0}^{\infty} b_i \tfrac{x^{n+i}}{(n+i)!}, \end{aligned}$$

see (4.2) and (4.4):

$$= f(x) - R_n(x) + S_n(x),$$

thus $h \in A$. Because of (4.3),(4.5) we have for $n = \max\{k, m\}$

$$\begin{aligned} \|f - h\| &= \|R_n - S_n\| \leq \|R_n\| + \|S_n\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

thus

$$h \in U_\varepsilon. \tag{4.7}$$

For the n-th derivation $D^n(h)$ of h we find with (1.3) and (4.6):

$$\begin{aligned} D^n(h)(x) &= D^n\left(\sum_{i=0}^{\infty} c_i \tfrac{x^i}{i!}\right) = \sum_{i=0}^{\infty} c_{n+i} \tfrac{x^i}{i!} = \sum_{i=0}^{\infty} b_i \tfrac{x^i}{i!} \\ &= g(x), \end{aligned}$$

thus $D^n(h) = g$.
With (4.7) we have proved proposition (4.1), thus the differential operator is topologically transitive (2.13).

qed.

Now we shall prove (2.16).

Proposition 2 *Let $f \in A, \varepsilon > 0$ and U_ε be the ε-neighborhood of f in the norm-topology. Then there exist $m \in \mathbb{N}$ and $g \in A$ with*

$$g \in \Pi_m \cap U_\varepsilon \tag{4.8}$$

with Π_m defined in (2.4).

Proof of (4.8)
The Taylor series

$$f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

is absolutely convergent for $|x| < R_f$ with $R_f > 1$. Therefore

$$K := \sum_{i=0}^{\infty} \frac{|a_i|}{i!} \geq \| \sum_{i=0}^{\infty} a_i \frac{x^i}{i!} \| \tag{4.9}$$

is finite and we can find $k_1 \in \mathbb{N}$ such that with the Euler number e

$$\frac{Ke}{n!} < \frac{\varepsilon}{2} \quad \text{for} \quad n \geq k_1. \tag{4.10}$$

Now we consider the sequence $(\|R_n\| \big| n \in \mathbb{N})$ of the remainders $R_n : E \to \mathbb{C}$ defined by

$$R_n(x) := \sum_{i=n}^{\infty} a_i \frac{x^i}{i!},$$

because it is a zero-sequence we can find $k_2 \in \mathbb{N}$ such that

$$\|R_n\| \leq \sum_{i=n}^{\infty} |a_i| \frac{1}{i!} < \frac{\varepsilon}{2} \quad \text{for each} \quad n \geq k_2. \tag{4.11}$$

By $m := \max\{k_1, k_2\}$ and the Taylor sequence $(a_i | i \in \mathbb{N}_0)$ we construct a function $g_m \in \Pi_m$, the sequence $(b_i \in \mathbb{C} \big| i \in \mathbb{N}_0)$ defined by

$$b_i := a_{i \bmod m} \qquad \text{for} \qquad i \in \mathbb{N}_0 \tag{4.12}$$

is periodic and bounded, and hence for the function $g_m : E \to \mathbb{C}$ defined by

$$g_m(x) := \sum_{i=0}^{\infty} b_i \frac{x^i}{i!} = \sum_{i=0}^{m-1} b_i \frac{x^i}{i!} + \sum_{i=m}^{\infty} b_i \frac{x^i}{i!}$$

we have with (2.10)

$$g_m \in A \qquad \text{and} \qquad g_m \in \Pi_m. \tag{4.13}$$

To finish the proof of Proposition 2 we show: $g \in U_\varepsilon$. With (4.12) we have

$$\begin{aligned} \|f-g\| &= \|\sum_{i=m}^{\infty} a_i \tfrac{x^i}{i!} - \sum_{i=m}^{\infty} b_i \tfrac{x^i}{i!}\| \le \|\sum_{i=m}^{\infty} a_i \tfrac{x^i}{i!}\| + \|\sum_{i=m}^{\infty} b_i \tfrac{x^i}{i!}\| \\ &\le \sum_{i=m}^{\infty} |a_i| \tfrac{1}{i!} + \sum_{i=m}^{\infty} |b_i| \tfrac{1}{i!}, \end{aligned}$$

and, because of (4.11) and $m \ge k_2$

$$< \tfrac{\varepsilon}{2} + \sum_{i=m}^{\infty} |b_i| \tfrac{1}{i!}.$$

From (4.12) we see

$$\begin{aligned} \sum_{i=m}^{\infty} |b_i| \tfrac{1}{i!} &= |a_0| \left(\tfrac{1}{m!} + \tfrac{1}{(2m)!} + \tfrac{1}{(3m)!} + \cdots\right) + |a_1| \left(\tfrac{1}{(m+1)!} + \tfrac{1}{(2m+1)!} + \tfrac{1}{(3m+1)!} + \cdots\right) + \cdots \\ &\quad + |a_{m-1}| \left(\tfrac{1}{(2m-1)!} + \tfrac{1}{(3m-1)!} + \tfrac{1}{(4m-1)!} + \cdots\right) \\ &= \tfrac{|a_0|}{m!} \left(1 + \tfrac{1}{(m+1)(m+2)\ldots(2m)} + \tfrac{1}{(m+1)(m+2)\ldots(3m)} + \cdots\right) \\ &\quad + \tfrac{|a_1|}{(m+1)!} \left(1 + \tfrac{1}{(m+2)(m+3)\ldots(2m+1)} + \tfrac{1}{(m+2)(m+3)\ldots(3m+1)} + \cdots\right) + \cdots \\ &\quad + \tfrac{|a_{m-1}|}{(2m-1)!} \left(1 + \tfrac{1}{2m(2m+1)\ldots(3m-1)} + \tfrac{1}{2m(2m+1)\ldots(4m-1)} + \cdots\right) \\ &\le \tfrac{|a_0|}{m!} \left(1 + \tfrac{1}{m!} + \tfrac{1}{(2m)!} + \cdots\right) + \tfrac{|a_1|}{(m+1)!} \left(1 + \tfrac{1}{m!} + \tfrac{1}{(2m)!} + \cdots\right) + \cdots \\ &\quad + \tfrac{|a_{m-1}|}{(2m-1)!} \left(1 + \tfrac{1}{m!} + \cdots\right) \\ &\le e \left(\tfrac{|a_0|}{m!} + \tfrac{|a_1|}{(m+1)!} + \cdots + \tfrac{|a_{m-1}|}{(2m-1)!}\right) \\ &= \tfrac{e}{m!} \left(|a_0| + \tfrac{|a_1|}{m+1} + \cdots + \tfrac{|a_{m-1}|}{(m+1)(m+2)\ldots(2m-1)}\right) \\ &\le \tfrac{e}{m!} \left(|a_0| + \tfrac{|a_1|}{1!} + \cdots + \tfrac{|a_{m-1}|}{(m-1)!}\right) \\ &\le \tfrac{e}{m!} \sum_{i=0}^{\infty} |a_i| \tfrac{1}{i!} \\ &= \tfrac{e}{m!} K < \tfrac{\varepsilon}{2} \end{aligned}$$

because $m \ge k_1$, see (4.9). Thus $g_m \in U_\varepsilon \cap \Pi_m$ and we have proved Proposition 2 and Theorem 4.

qed.

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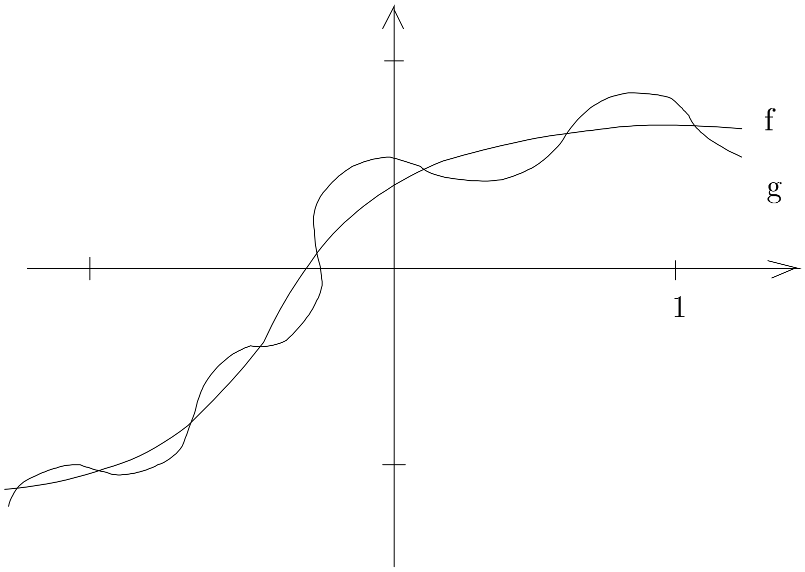

Figure 1: Graphical representation for $x \in [-1, 1]$ of function f and g of Theorem 1

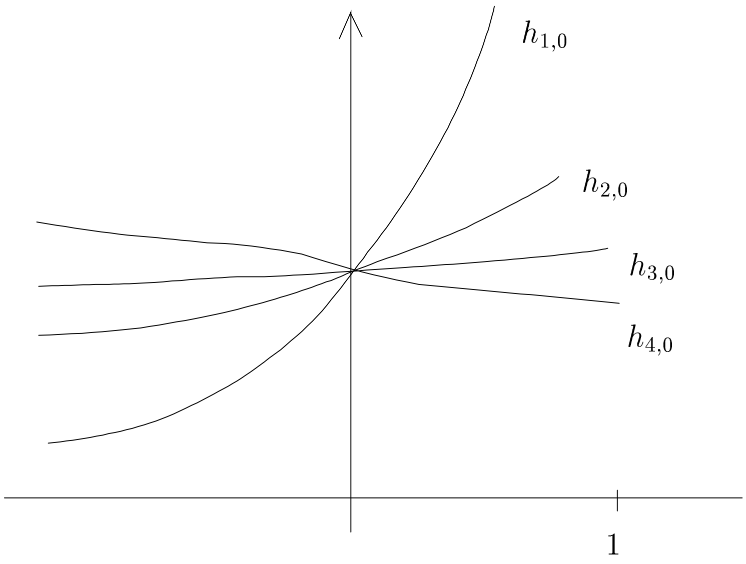

Figure 2: Graphical representation of the hyperbolic functions $h_{1,0}, h_{2,0}, h_{3,0}, h_{4,0}$

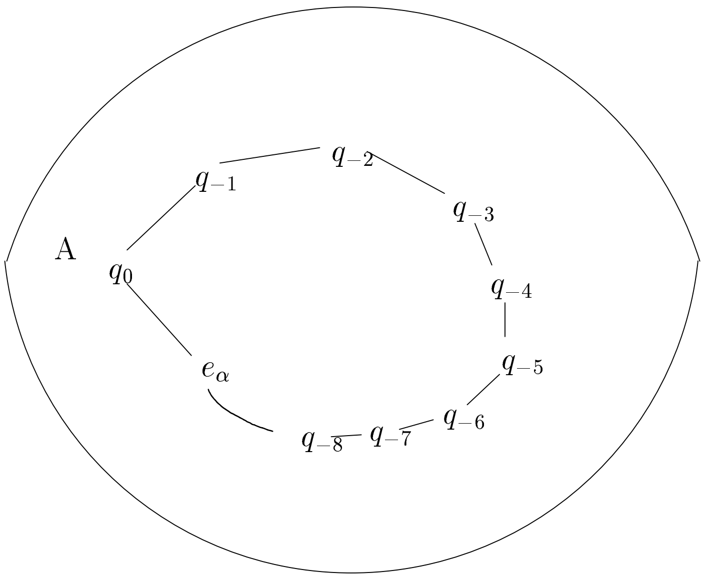

Figure 3: An orbit $\{q_i \in A | i \in \mathbb{Z}\}$ that is homoclinic to the fixed element e_α

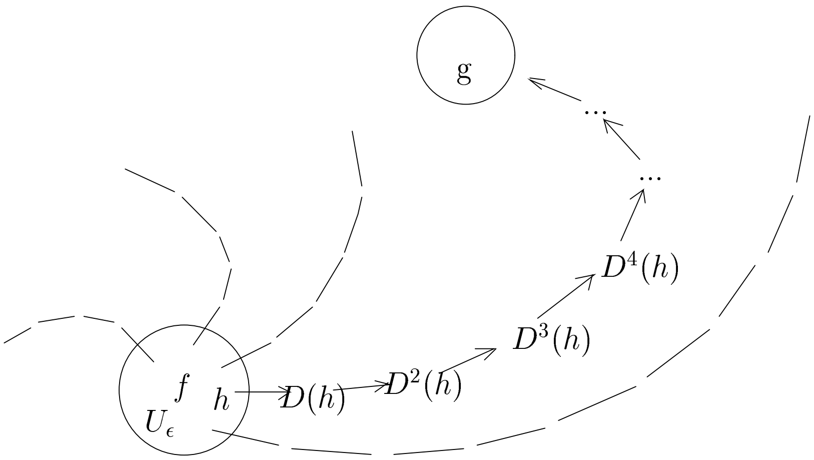

Figure 4: Topological transitivity (mixing property) of the differential operator

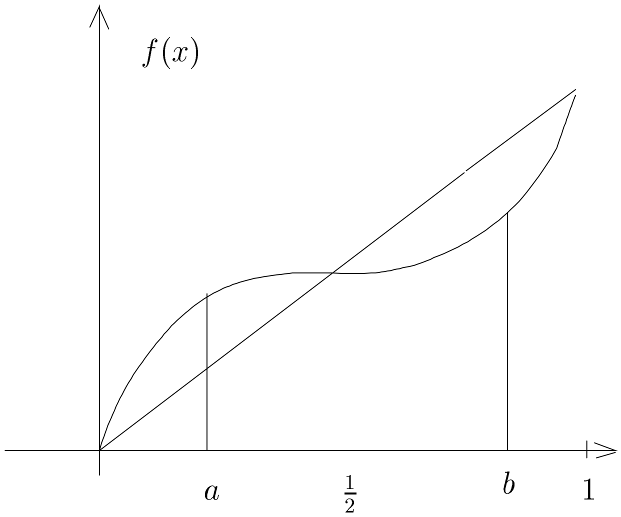

Figure 5: An discontinuous function which is not sensitiv, compare (2.17)