SLICES IN THE UNIT BALL OF A UNIFORM ALGEBRA

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Abstract. We show that every nonvoid relatively weakly open subset, in particular every slice, of the unit ball of an infinite-dimensional uniform algebra has diameter 2.

It is an important task in Banach space theory to determine the extreme point structure of the unit ball for various examples of Banach spaces. The most common way to describe "corners" of convex sets is by looking for *extreme points, exposed points, denting points* and *strongly exposed points.*

Let C be a closed, bounded subset of a Banach space Y. A *slice* of C is a set of the form

$$S(y^*,\varepsilon,C)=\{y\in C\colon\ \operatorname{Re}y^*(y)\geq\sup\operatorname{Re}y^*(C)-\varepsilon\},$$

where $y^*\in Y^*$. Let $B_\varepsilon(y)$ denote the ball with radius ε centred at y. A *denting point* of C is a point y_0 in C such that $y_0\notin\overline{\mathrm{co}}(C\setminus B_\varepsilon(y_0))$ for every $\varepsilon>0$. Thus, by the Hahn-Banach theorem, if $y_0\in C$ is a denting point, then there are slices of C of arbitrarily small diameter containing y_0. When C has slices of arbitrarily small diameter it is called *dentable.*

For the definitions of an extreme point, an exposed point and a strongly exposed point, we refer to [2]. More or less directly from the definitions it follows that every strongly exposed point is both denting and exposed, and every denting or exposed point is extreme.

In this note $\mathcal{A}$ denotes an infinite-dimensional uniform algebra, i.e., an infinite-dimensional closed subalgebra of some $C(K)$-space which separates the points of K and contains the constant functions. In [1] Beneker and Wiegerinck demonstrated the non-existence of strongly exposed points in $B_{\mathcal{A}}$, the closed unit ball of $\mathcal{A}$. Here we shall prove a stronger result by more elementary means. A corollary of our result is that the set of denting points is, in fact, also empty.

The set of extreme points and the set of exposed points of the unit ball have been characterised in many uniform algebras. For example, there are lots of such points in the unit ball of H^∞, the algebra of bounded analytic functions on the unit disk, since every inner function is an exposed point (see [3, p. 221]).

Let $\mathcal{A}$ be a function algebra on a compact space K. A point $x\in K$ is called a *strong boundary point* if for every neighbourhood V of x and every $\delta>0$, there is some $f\in\mathcal{A}$ such that $f(x)=\|f\|=1$ and $|f|\leq\delta$ off V. The closure of the set of strong boundary points is the *Silov boundary* of $\mathcal{A}$. It is a fundamental result in the theory of uniform algebras that one can identify

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$\mathcal{A}$ as a uniform algebra on its Silov boundary; cf. [8, p. 48 and p. 78]. In the sequel we shall therefore assume that the set of strong boundary points is dense in K.

We now turn to our first result, which gives a quantitative statement of the non-dentability of $B_{\mathcal{A}}$.

Theorem 1. *Every slice of the unit ball of an infinite-dimensional uniform algebra $\mathcal{A}$ has diameter 2.*

Proof. Take an arbitrary slice $S = \{a \in B_{\mathcal{A}}\colon \operatorname{Re} \ell(a) \geq 1-\varepsilon\}$, where $\|\ell\| = 1$. We will produce two functions in S having distance nearly 2.

Let $0 < \delta \leq \varepsilon/11$. We first pick some $f \in B_{\mathcal{A}}$ such that

$$\operatorname{Re} \ell(f) \geq 1 - \delta.$$

The functional ℓ can be represented by a regular Borel measure μ on K with $\|\mu\| = 1$, i.e., $\ell(a) = \int_K a\,d\mu$ for all $a \in \mathcal{A}$. Let $\emptyset \neq V_0 \subset K$ be an open set with $|\mu|(V_0) \leq \delta$; such a set exists since K is infinite. Fix a strong boundary point $x_0 \in V_0$. Using the definition of a strong boundary point, inductively construct functions $g_1, g_2, \ldots \in \mathcal{A}$ and nonvoid open subsets $V_0 \supset V_1 \supset V_2 \supset \ldots$ such that

$$g_n(x_0) = \|g_n\| = 1, \quad |g_n| \leq \delta \text{ on } K \setminus V_{n-1}$$

and

$$V_n = \{x \in V_{n-1}\colon |g_n(x) - 1| < \delta\}.$$

Let $N > 1/\delta$ and define

$$g = \frac{1}{N}\sum_{k=1}^{N} g_k, \quad h = f(1-g) \in \mathcal{A}.$$

By construction, $|h| \leq \delta$ on V_N and $|h| \leq 1+\delta$ on $K \setminus V_0$. We claim that $\|h\| \leq 1 + 3\delta$. In fact, if $x \in V_{r-1} \setminus V_r$, then $|1 - g_k(x)| \leq \delta$ if $1 \leq k < r$, $|g_r(x)| \leq 1$ and $|g_k(x)| \leq \delta$ if $r < k \leq N$, and therefore

$$|h(x)| \leq \frac{1}{N}\sum_{k=1}^{N} |1 - g_k(x)| \leq \frac{(N-1)(1+\delta)+2}{N} \leq 1 + 3\delta.$$

We now estimate $|\ell(f) - \ell(h)|$:

$$\begin{aligned}
|\ell(f) - \ell(h)| &\leq \int_{K\setminus V_0} |f-h|\,d|\mu| + \int_{V_0} |f-h|\,d|\mu| \\
&\leq \int_{K\setminus V_0} |g|\,d|\mu| + \int_{V_0} (|f| + |h|)\,d|\mu| \\
&\leq \delta + (2+3\delta)|\mu|(V_0) \leq 4\delta.
\end{aligned}$$

Next, we produce a function $\varphi \in \mathcal{A}$ such that

$$\varphi(x_0) = \|\varphi\| = 1, \quad |\varphi| \leq \delta \text{ on } K \setminus V_N.$$

We then have $\|h \pm \varphi\| \leq 1 + 4\delta$, and the functions $\psi_\pm = (h \pm \varphi)/(1+4\delta)$ are in the unit ball of $\mathcal{A}$. We have $|\ell(\varphi)| \leq 2\delta$ and thus

$$|\ell(\psi_\pm) - \ell(h)| \leq |\ell(h)|\frac{4\delta}{1+4\delta} + \frac{2\delta}{1+4\delta} \leq 6\delta.$$

Consequently,

$$\operatorname{Re}\ell(\psi_\pm) \geq \operatorname{Re}\ell(f) - 10\delta \geq 1 - 11\delta \geq 1 - \varepsilon$$

so that $\psi_\pm \in S$; but $\|\psi_+ - \psi_-\| = 2/(1+4\delta) \to 2$ as $\delta \to 0$. Hence $\operatorname{diam} S = 2$. □

The point of working with g rather than g_1 in the proof is to control $\|1-g\|$. Another way to achieve this is to construct a suitable conformal map ϕ from the unit disk to a neighbourhood of $[0,1]$ in $\mathbb{C}$ and to consider $\phi \circ g_1$.

We now extend Theorem 1 to relatively weakly open subsets.

Theorem 2. *Every nonvoid relatively weakly open subset W of the unit ball of an infinite-dimensional uniform algebra $\mathcal{A}$ has diameter* 2.

Proof. Every nonvoid relatively weakly open subset of the unit ball of a Banach space contains a convex combination of slices, see [4, Lemma II.1] or [12]. Thus, if $W \subset B_{\mathcal{A}}$ is given as above, there are slices $S^{(1)}, \dots, S^{(n)}$ and $0 \leq \lambda_j \leq 1$, $\sum_{j=1}^n \lambda_j = 1$, such that $\sum_{j=1}^n \lambda_j S^{(j)} \subset W$.

Let $S^{(j)} = \{a \in B_{\mathcal{A}}\colon \operatorname{Re}\ell_j(a) \geq 1 - \varepsilon_j\}$ with $\|\ell_j\| = 1$ and representing measures μ_j. We now perform the construction of the proof of Theorem 1 with $\varepsilon = \min \varepsilon_j$, $0 < \delta \leq \varepsilon/11$ as before and a nonvoid open set $V_0 \subset K$ such that $|\mu_j|(V_0) \leq \delta$ for all j. We obtain functions $h^{(j)}$ and φ (independently of j) such that $(h^{(j)} \pm \varphi)/(1+4\delta) \in S^{(j)}$ and $\|\varphi\| = 1$. Therefore $\sum_{j=1}^n \lambda_j h^{(j)} \pm \varphi \in (1+4\delta)W$, and $\operatorname{diam} W = 2$. □

In case K does not have isolated points, Theorem 1 is a formal consequence of the Daugavet property of $\mathcal{A}$, proved in [14] or [13], and [7, Lemma 2.1]. Likewise Theorem 2 follows from [12] in this case. We are grateful to K. Jarosz for pointing out to us that Theorem 2 can also be deduced from results of his in [6].

It is clear from the description in terms of slices that a denting point of the unit ball of a Banach space Y is a point of continuity for the identity mapping from (B_Y, weak) to (B_Y, norm). Conversely, it is known that a point of continuity which is an extreme point of B_Y is a denting point [9], [10].

The following corollary is immediate from Theorem 2.

Corollary 3. *The unit ball of an infinite-dimensional uniform algebra does not contain any denting points or merely points of continuity for the identity mapping with respect to the weak and the norm topology.*

A related result we would like to mention is the theorem due to Hu and Smith stating that the unit ball in the space of continuous functions from a compact Hausdorff space into a Banach space equipped with its weak topology has no denting points [5]. This result was also obtained by T.S.S.R.K. Rao [11], who has informed us that he has also given a proof of Corollary 3 based on techniques from that paper.

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