Some Constructions of Divisible Designs from Laguerre Geometries

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Abstract

In the nineties, A.G. Spera introduced a construction principle for divisible designs. Using this method we get series of divisible designs from finite Laguerre geometries. We show a close connection between these divisible designs and divisible designs whose construction started from a conic in a plane of a 3-dimensional projective space.

1 Introduction

There exist a lot of different kinds of designs. One interesting sort are divisible designs. In 1992, A. G. Spera introduced a method to construct them by $t-R$-homogeneous ($t-R$-transitive) R-permutation groups ([9]). There are already several examples of constructions using Spera's construction principle ([9], [8], [4],[3]). In 1999, R.-H. Schulz and C. Cerroni started their construction from a conic in a plane of the 3-dimensional projective space $\mathrm{PG}(3,q)$ ([4]).

In this paper we construct series of divisible designs also by using this method but starting from a Laguerre geometry $\Sigma(\mathrm{GF}(q), \mathbb{D}(\mathrm{GF}(q)))$ with $\mathbb{D}(\mathrm{GF}(q))$ the ring of dual numbers over the finite field $\mathrm{GF}(q)$. We show a close connection between these designs and those, which are constructed by Schulz and Cerroni.

2 Preliminary terms, definitions and results

2.1 About Divisible Designs

Let X be a finite set with an equivalence relation R on its elements. We denote by $[x]$ the R-equivalence class containing $x \in X$ and define $S := \{[x] | x \in X\}$.

Definition 2.1.1 A subset Y of X is called *R-transversal*, if $|Y \cap [x]| \leq 1$ for all $x \in X$.

Definition 2.1.2 (Definition of divisible designs) Let t, s, k, λ_t be positive integers with

$$t \leq k < v = |X|.$$

A triple $\mathcal{D} = (X, \mathcal{B}, S)$ is called $t-(s,k,\lambda_t)$-*divisible design* (or $t-(s,k,\lambda_t)$-DD) if

(1) $\mathcal{B}$ is a set of R-transversal subsets of X with $|B| = k$ for all $B \in \mathcal{B}$;
(2) $|[x]| = s$ for all $x \in X$;
(3) for every R-transversal t-subset Y of X, there exist exactly λ_t elements of $\mathcal{B}$ containing Y.

The elements of X are called *points*, those of $\mathcal{B}$ *blocks*, and the elements of S *point classes*.

Definition 2.1.3 A divisible design is called *transversal* if each block meets exactly each point class, otherwise it is called *regular*.

In this paper we always suppose, that every divisible design is *simple*, that means, there exist no repeated blocks. Note that, for $t \geq 3$, a t- divisible design is also a $(t-1)$-divisible design with $\lambda_{t-1} = \lambda_t(v - st + s)(k - t + 1)^{-1}$.
One possibility to construct divisible designs is given by the following proposition of A.G. Spera.

2.1.1 SPERA's construction principle

Definition 2.1.4 Let G be a group operating on the set X and R an equivalence relation on X which is G-admissible, that is

$$xRy \Longrightarrow x^g R y^g (\text{ for all } g \in G, \;\; x, y \in X);$$

then $\Lambda = (G, X, R)$ is called an $R-$*group*.
(The group G induces a permutation group on X, but not necessarily faithfully.)

Definition 2.1.5 Λ is called $t - R$-*transitive* if for any two R-transversal t-tuples $(x_1, \dots, x_t)$ and $(y_1, \dots, y_t)$ of different elements of X, there exists an element g of G such that $y_i = x_i^g$ for $i = 1, 2, \dots, t$.

Proposition 2.1.1 (A.G. Spera, [9]) *Let $\Lambda = (G, X, R)$ be a finite $t - R$-transitive R-group, and let B be an R-transversal subset of X with $t \leq k := |B| < v := |X|$. Then the incidence structure*

$$D(\Lambda, B) = (X, B^G, S) \text{ for } B^G = \{B^g \mid g \in G\},$$

is a $t-(s,k,\lambda_t)$-divisible design with $s = |[x]|$ for some $x \in X$, $k = |B|$, $b = \frac{|G|}{|G_B|}$ and $\lambda_t = |G| \begin{pmatrix} k \\ t \end{pmatrix} \left(|G_B| \begin{pmatrix} \frac{v}{s} \\ t \end{pmatrix} s^t \right)^{-1}$, where G_B denotes the setwise stabilizer of B and b the number of blocks of $D(\Lambda, B)$. Moreover, G induces a point- and block-transitive automorphism group of $D(\Lambda, B)$.

Remark 2.1.1 To construct divisible designs by using Spera's proposition we need a finite set X with an equivalence relation on its elements and a finite t-R-transitive R-permutation group operating on this set. Then we have to choose a so called 'base block' and to calculate the parameters.

2.2 Classical Laguerre Geometry

In the classical Laguerre geometry we deal with so called spears and cycles. A spear is an oriented line and a cycle is either an oriented circle or a point in the real plane. Besides the relation of contact between the spears and cycles there exists an equivalence relation on the spears - the parallelism relation. Two spears are called parallel if their lines are parallel and the orientations are equal. One possibility to represent spears and cycles in the 3-dimensional real space is given by Blaschke's Cylinder-Model ([2]). In this model every spear corresponds to exactly one point of a given cylinder C over an ellipse in the 3-dimensional real space. The cycles are represented by the intersections of C with planes which are not parallel to the axis of C.
Two spears are parallel iff the corresponding points lie on the same generator of C ([1]). So there exists an equivalence relation on the points of C, induced by the parallelism relation on the spears.
For another representation of the spears, we need the ring of dual numbers over $\mathbb{R}$. There exists an isomorphism between the spears and the points of the projective line over the ring of dual numbers over $\mathbb{R}$, which preserves the parallelism relation ([1]).
For our construction of divisible designs, we have a look at a generalization of the classical Laguerre geometry, the Laguerre geometry over the finite field $\mathrm{GF}(q)$.

2.3 A Laguerre Geometry over the finite field GF(q)

Definition 2.3.1 A *dual number over the finite field GF*(q) is an ordered pair (a, b) with $a, b \in \mathrm{GF}(q)$ and with the following properties.

Two dual numbers are equal if their components are equal. The rules for addition and multiplication are:

(i) $(a, b) + (a', b') := (a + a', b + b')$

(ii) $(a, b) \cdot (a', b') := (aa', ab' + ba')$

A dual number (a, b) can also be represented either by a matrix $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathrm{GF}(q)$ or in the following form: $a + b\epsilon$ with $\epsilon^2 = 0$. The rules for addition and multiplication are corresponding to those for matrices.

Remark 2.3.1 As a subring of the ring of matrices, the set of dual numbers with the given addition and multiplication is a ring with 1. We denote this ring by $\mathbb{D}(\mathrm{GF}(q))$.

In analogy to Benz ([1], p.24) we have

Proposition 2.3.1 $\mathbb{D}(\mathrm{GF}(q))$ *is an abelian local ring.*

The only maximal ideal N contains all noninvertible elements of $\mathbb{D}(\mathrm{GF}(q))$. We define $\mathcal{R} := \mathbb{D}(\mathrm{GF}(q))^* = \mathbb{D}(\mathrm{GF}(q)) - N$ as the group of all invertible elements.

Definition 2.3.2 (Laguerre algebra) For K a field, a K-algebra A is called a *Laguerre algebra* provided there is a two-sided ideal M of A with $A^* = A \setminus M$ and $A = K \oplus M$.

$\mathbb{D}(\mathrm{GF}(q))$ is abelian, hence the above defined ideal N is two-sided. The direct sum of the embedded field $\mathrm{GF}(q)$ and N is $\mathbb{D}(\mathrm{GF}(q))$, hence we get

Proposition 2.3.2 *$\mathbb{D}(\mathrm{GF}(q))$ is a Laguerre algebra.*

Definition 2.3.3 We define the *projective line* $\mathbb{P}(\mathbb{D}(\mathrm{GF}(q)))$ *over* $\mathbb{D}(\mathrm{GF}(q))$ as the set of all equivalence classes of admissible pairs. Here, we call a pair (x_1, x_2), $x_1, x_2 \in \mathbb{D}(\mathrm{GF}(q))$, an *admissible pair over* $\mathbb{D}(\mathrm{GF}(q))$ if at least one element is invertible. Two admissible pairs $(x_1, x_2), (y_1, y_2)$ are called *equivalent* if there exists an element $r \in \mathcal{R}$ such that $x_i = ry_i$, $i = 1, 2$. We call the elements of $\mathbb{P}(\mathbb{D}(\mathrm{GF}(q)))$ *points.*
Since $\mathbb{D}(\mathrm{GF}(q))$ is a local ring, this definition of the projective line over $\mathbb{D}(\mathrm{GF}(q))$ is equivalent to that, given by Herzer [5] on p. 785.

Definition 2.3.4 Two points $P, Q \in \mathbb{P}(\mathbb{D}(\mathrm{GF}(q)))$ with $P = \mathcal{R}(p_1, p_2), Q = \mathcal{R}(q_1, q_2)$, $p_i, q_i \in \mathbb{D}(\mathrm{GF}(q))$, $i = 1, 2$ are called *parallel* if

$$p_1q_2 - q_1p_2 = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \notin \mathcal{R}.$$

In analogy to the more general definition of a chain geometry of Benz in ([1], p. 94), we define, in this paper, the chain geometry $\Sigma(K, D)$ with $K := \mathrm{GF}(q), D := \mathbb{D}(\mathrm{GF}(q))$ as an incidence structure whose points are the elements of $\mathbb{P}(D)$ and whose blocks (chains) are the images of $\mathbb{P}(K)$ under the projective group of $\mathbb{P}(D)$ (cf. Def. 2.3.5 and [5], p. 790).
Since D is a Laguerre algebra, the following holds:

Proposition 2.3.3 ([5]) *$\Sigma(K, D)$ is a so called* Laguerre geometry, *i.e. the parallelism relation is an equivalence relation on $\mathbb{P}(D)$ and every chain of $\Sigma(K, D)$ meets every parallel class of points.*

Now consider the chain geometry $\Sigma(K, D)$ whose points are the elements of the projective line over D. We can divide the points of $\mathbb{P}(D)$ into *proper* and *improper* points, depending on the invertiblity of the second component or, equivalently, on the parallelism to the point $\mathcal{R}(1, 0)$. Every proper point can be represented as $\mathcal{R}(p, 1)$, $p \in D$ and every improper point as $\mathcal{R}(1, \delta\epsilon)$, $\delta \in K$.

Proposition 2.3.4 *(i) Let $P, Q \in \mathbb{P}(D)$ be two proper points with $P = \mathcal{R}(p_1 + p_2\epsilon, 1)$ and $Q = \mathcal{R}(q_1 + q_2\epsilon, 1)$. They are parallel iff $p_1 = q_1$.*

(ii) Improper points are always parallel.

(iii) A proper point is never parallel to an improper point.

Proof: (i) $P||Q \Leftrightarrow \begin{vmatrix} p_1 + p_2\epsilon & 1 \\ q_1 + q_2\epsilon & 1 \end{vmatrix} = p_1 - q_1 + (p_2 - q_2)\epsilon \notin \mathcal{R} \Leftrightarrow p_1 - q_1 = 0 \Leftrightarrow$ $p_1 = q_1$. (ii),(iii) follow by easy calculations. $\square$

Remark 2.3.2 With the parallelism relation we get $q+1$ equivalence classes with q elements each: $\{\mathcal{R}(x+b\epsilon,1)|b \in K\}$, with $x \in K$ and $[\mathcal{R}(1,0)] = \{\mathcal{R}(1,\delta\epsilon)|\delta \in K\}$.

Remark 2.3.3 We can embed the projective line over K into $\mathbb{P}(D)$. The elements of $\mathbb{P}(K)$ form the following transversal subset of $\mathbb{P}(D)$: $\tilde{K} := \{\mathcal{R}(p_1 + 0\epsilon, 1)|p_1 \in K\} \cup \mathcal{R}(1,0)$.

Definition 2.3.5 We define the projective group of $\mathbb{P}(D)$ as the group of all regular 2×2 matrices with entries of $\mathbb{D}(K)$, factorised by the subgroup $\{\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right) | r \in \mathcal{R}\}$. We denote it by $\Gamma(D)$. (cf. [1])

Proposition 2.3.5 ([1]) $\Gamma(D)$ *acts sharply 3-R-transitively on the point set of* $\mathbb{P}(D)$ *and respects the parallelism relation.*

By remark 2.1.1 we are now able to construct a divisible design with proposition 2.1.1.
Before we start the construction, we have a look at a possible representation of $\Sigma(K,D)$ in a 3-dimensional projective space, analogous to Blaschke's Cylinder-Model ([2]).

Theorem 2.3.1 (Hotje, [7], p.9) *Let L be an abelian kinematic algebra of rank* $n \geq 2$ *over the field F and M the corresponding quadratic form. Then the chain geometry* $\Sigma(F,L)$ *can be represented in the (n+1)-dimensional projective space* $\Psi = (L \times F \times F)^*/F^*$ *over F, as the set of those points of a quadric* $\mathcal{O} = \{[a,c,d]|M(a) = cd\}$ *which are not in the radical of the corresponding bilinear form. The chains are exactly the intersections of* $\mathcal{O}$ *with those planes of* Ψ*, which do not have a whole line, but at least 3 points in common with* $\mathcal{O}$*.*

The proof can be found in ([7]).
For every dual number[1] $x = x_1 + x_2\epsilon$, there exist elements $k_x = -2x_1$, $l_x = x_1^2$ of GF(q), such that $x^2 = -l_x - k_x x$ holds. Therefore we get

Lemma 2.3.1 *D is an abelian kinematic algebra over K.*

Since D is of rank 2 we know with the theorem of H. Hotje, that we can embed $\Sigma(K,D)$ in a 3-dimensional projective space such that every point of $\mathbb{P}(D)$ is mapped onto a point of a quadric $\mathcal{O}$. We will determine this quadric analogously to the more general case showed by Hotje ([7]). For that we have to define the involution

$$'\_' : \left\{ \begin{array}{ll} D & \longrightarrow D \\ x & \longmapsto \bar{x} = x_1 - x_2\epsilon, \end{array} \right.$$

the quadratic form

$$M : \left\{ \begin{array}{ll} D & \longrightarrow K \\ x & \longmapsto x\bar{x} = x_1^2 \end{array} \right.$$

[1] In the following paragraph a dual number x always stands for the precise expression $x = x_1 + x_2\epsilon$, $x_1, x_2 \in$ GF(q).

as well as the quadratic form

$$Q : \begin{cases} D \times K \times K & \longrightarrow K \\ (x, c, d) & \longmapsto M(x) - cd = x_1^2 - cd \end{cases}$$

with the corresponding bilinear form

$$S_Q : \begin{cases} (D \times K \times K)^2 & \longrightarrow K \\ (x, c, d) \times (y, u, v) & \longmapsto 2x_1 y_1 - cv - ud. \end{cases}$$

We embed $\mathbb{P}(D)$ in the 3-dimensional projective space Ψ, with $\Psi := (D \times K \times K)^* / K^*$: The expression $[a, a_1, a_2]$ describes the point of Ψ, which is determined by $(a, a_1, a_2) \in (D \times K \times K)^*$, then

$$\Delta : \begin{cases} \mathbb{P}(D) & \longrightarrow \Psi \\ \mathcal{R}(x, y) & \longmapsto [x\bar{y}, M(x), M(y)] \end{cases}$$

is an injective mapping (cf. [7]).

We define $\mathcal{O} := \{[x, c, d] \in \Psi | Q((x, c, d)) = 0\}$ as a cone in Ψ. The vertex of this cone consists of a single point, say Rad Ψ, since the radical of the quadratic form Q is a one-dimensional subspace of $D \times K \times K$. We call a plane of Ψ *admissible* if its intersection with the quadric $\mathcal{O}$ is the image of a chain of $\mathbb{P}(D)$.

Lemma 2.3.2 (cf.[7],(3.2), p.6, (3.5) and a special case of (3.6), p.9)

(i) $\Delta(\mathbb{P}(D)) = \mathcal{O} \setminus \text{Rad } \Psi$.

(ii) Let $P, Q \in \mathbb{P}(D), P \neq Q$.

$$P || Q \iff \Delta(P) \text{ and } \Delta(Q) \text{ are lying on the same generator}^2 \text{ of } \mathcal{O}.$$

(iii) The admissible planes are exactly those planes of Ψ, which do not pass through the vertex Rad Ψ.

Now we have a look at $\Delta(\tilde{K}) = \{[p_1, p_1^2, 1] | p_1 \in K\} \cup \{[0, 1, 0]\}$ (cf. remark 2.3.3) which is a regular conic, and define $\tilde{W}$ as the (admissible) plane of Ψ which is determined by $[0, 1, 0], [1, 1, 1], [0, 0, 1] \in \Delta(\tilde{K})$. Hotje showed

Lemma 2.3.3 ([7], (3.4), p.8) $\Delta(\tilde{K}) = \mathcal{O} \cap \tilde{W}$.

Remark 2.3.4 We can identify the elements of $\mathbb{P}(D)$ with the elements of the cone $\mathcal{O}$, except its vertex $E := [a, 0, 0]$, in the 3-dimensional projective space Ψ. Consider a plane in Ψ whose intersection with $\mathcal{O}$ is exactly the point E, then all points of the cylinder $Z := \mathcal{O} \setminus E$ are affine points of the 3-dimensional affine space whose ideal plane is the plane considered above. Such a plane exists, since no finite field is quadratically closed. Z consists of $q + 1$ lines (generators) with q points each, which intersect the ideal plane in E. By lemma 2.3.2 points on the same line are mutually parallel.
Notice, that $\mathbb{P}(K)$ is embedded in $Z \cap \tilde{W}$ (lemma 2.3.3).

[2] A generator of $\mathcal{O}$ is a line which is completely contained in $\mathcal{O}$.

3 Construction of divisible designs

3.1 - from a Laguerre geometry

Theorem 3.1.1 *Let $n, i \in \mathbb{N}$ with $i|n$ and let $q = p^n$, where p is a prime number. Then there exist $3-(q,k,\lambda_3)$-divisible designs, each with $q(q+1)$ points, with the parameters k and λ_3 given in Table 1, where p and i are subject to the conditions given there. These 3-divisible designs admit $\Gamma(\mathbb{D}(\mathrm{GF}(q)))$ as a point- and block-transitive automorphism group. The same holds for the corresponding 2-divisible designs.*

k	λ_3	Conditions
p^i+1	1	
p^i	p^i-2	$p^i > 2$
p^i-1	$\frac{1}{2}(p^i-2)(p^i-3))$	$p^i > 3$
p^i-2	$\frac{1}{6}(p^i-2)(p^i-3)(p^i-4))$	$p^i > 4$
p^i-3	$\frac{1}{4}(p^i-3)(p^i-4)(p^i-5))$	$p^i > 7$
4	6	
p^i-3	$\frac{1}{24}(p^i-3)(p^i-4)(p^i-5))$	$p = 3$ and $p^i > 5$
4	1	
p^i-3	$\frac{1}{8}(p^i-3)(p^i-4)(p^i-5))$	$p > 3$ and $p^i > 5$
4	3	
p^i-3	$\frac{1}{12}(p^i-3)(p^i-4)(p^i-5))$	$p^i \equiv 1 \mod 3$ and $p^i > 5$
4	2	

Table 1

Proof of theorem 3.1.1:

Consider the chain geometry $\Sigma(\mathrm{GF}(q), \mathbb{D}(\mathrm{GF}(q)))$, $q = p^n$, where p is a prime number. We use the same notation as above. By prop. 2.3.5 we have, with $\Gamma(D)$, a 3-R-transitive R-permutation group operating on the point set, which consists of the q^2 proper points and q improper points. They are divided into $q+1$ parallel classes with q elements each, giving the points and the point classes of a DD. For the convenience of the reader we show the calculation of $|\Gamma(D)|$ and of the order of the stabilizer of the base block considered.

We calculate $|\Gamma(D)|$ by using the sharp $3-R$-transitivity and determining the order of the orbit of a transversal triple. Thus we get $|\Gamma(D)| = q^4(q^2-1)$.

Now we have to choose a base block. Let $i \in \mathbb{N}$ with $i|n$ where $q = p^n$.

(i) For $L := \mathrm{GF}(p^i)$ we embed the projective line over L

$$\mathbb{P}(L) = \mathrm{PG}(1, p^i) =: B$$

into the projective line over D and define it as our base block.
Notice, that a projectivity of $\Gamma(D)$ which maps three different points of B onto points of B belongs to $\Gamma(L)$ (cf. [5], Prop. 2.3.1, p.790). $\Gamma(L)$ operates 3-regularly on $\mathrm{PG}(1, p^i)$ and therefore (regarded as a subset of $\Gamma(D)$) on

B, too. Hence, the order of the stabilizer of B is $|\Gamma(L)| = p^i(p^{2i}-1)$.
In $\Sigma(L,D)$, three mutually nonparallel points are incident with exactly one chain (cf. [1], Theorem 1.1, p. 95), and the chains are exactly the blocks of our divisible design. Therefore we get by prop. 2.1.1 a $3-(s,k,\lambda_3)$-DD with $s=q$, $k:=|B|=p^i+1$ and $\lambda_3=1$.

This is also a 2-$(q, p^i+1, \frac{q(q-1)}{p^i-1})$-divisible design.

To construct the DD's of the cases (ii)-(v), we define $B' := B \setminus M$ as our base block by removing a set $M \subset B$. The block B' should contain at least three points, therefore p^i has to be big enough.
The stabilizer of B' in $\Gamma(D)$ has to be in $\Gamma(L)$ (see above) and thus to fix B and hence also M setwise. Therefore

$$\Gamma(D)_{B'} = \Gamma(L)_M.$$

Notice that, vice versa, the stabilizer of M in $\Gamma(D)$ is equal to the stabilizer of B' in $\Gamma(D)$ iff M consists of at least three elements.

(ii) Let $B' := B \setminus \{\mathcal{R}(1,0)\}$. This is a L-chain minus one point. $\Gamma(L)$ acts 3-regularly on B, hence $|\Gamma(L)| = |\Gamma(L)_{\mathcal{R}(1,0)}|(p^i+1)$ and therefore $|\Gamma(L)_{B'}| = p^i(p^i-1)$.
By proposition 2.1.1 we get a $3-(s,k,\lambda_3)$-divisible design with $s=q$, $k:=|B'|=p^i$ and $\lambda_3 = \frac{|\Gamma|}{|\Gamma_{B'}|}\begin{pmatrix} p^i \\ 3 \end{pmatrix} / \left[\begin{pmatrix} \frac{q^2+q}{q} \\ 3 \end{pmatrix} q^3\right] = p^i-2$.

This is also a 2-$(q,p^i,q(q-1))$-divisible design.

(iii) Let $B' := B \setminus \{\mathcal{R}(1,0), \mathcal{R}(0,1)\}$.
Concluding from the 3-regularity of $\Gamma(L)$ and the number of possible permutations of the elements of M, we know $|\Gamma(L)_{\{\mathcal{R}(1,0),\mathcal{R}(0,1)\}}| = 2(p^i-1)$. Therefore, by proposition 2.1.1 we get a $3-(s,k,\lambda_3)$-divisible design with $s=q$, $k:=|B'|=p^i-1$ and $\lambda_3 = \frac{1}{2}(p^i-2)(p^i-3)$.

This is also a 2-$(q,p^i-1,\frac{1}{2}(p^i-2)q(q-1))$-divisible design.

(iv) Let $B' := B \setminus \{\mathcal{R}(1,0), \mathcal{R}(0,1), \mathcal{R}(1,1)\}$.
Similar to (iii) we can conclude $|\Gamma(L)_{\{\mathcal{R}(1,0),\mathcal{R}(0,1),\mathcal{R}(1,1)\}}| = 6$ from the 3-regularity of $\Gamma(L)$ and the fact, that there exist $3! = 6$ possible permutations of the elements of M. Now, we get a $3-(s,k,\lambda_3)$-divisible design with $s=q$, $k:=|B'|=p^i-2$ and $\lambda_3=\frac{1}{6}(p^i-2)(p^i-3)(p^i-4)$.

This is also a 2-$(q,p^i-2,\frac{1}{6}(p^i-2)(p^i-3)q(q-1))$-divisible design.

(v) Let $B' := B \setminus \{\mathcal{R}(1,0), \mathcal{R}(0,1), \mathcal{R}(1,1), \mathcal{R}(x,1)\}$, with $x \in L \setminus \{0,1\}$.
Analogous to the cases above the stabilizer of B' in $\Gamma(L)$ corresponds to the stabilizer of M in $\Gamma(L)$. Two 4-tuples of points are projectively equivalent iff their cross-ratios are equal. The four points of M allow 24 permutations, but they determine only the following 6 cross-ratios.

$$x,\ 1/x,\ 1-x,\ 1/(1-x),\ (x-1)/x,\ x/(x-1) \qquad (*)$$

In any case the cross-ratio of the four points is invariant under a projective group of order 4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ ([6], p.119/120).

(a) If all six values of $(*)$ are different, then the stabilizer of B' consists only of these four elements, since $24 = 6 \cdot 4$. This case occurs if the four points are neither harmonic nor equianharmonic.
If p is even, no harmonic quadruple exists, so x can be any element of $\mathrm{GF}(p^i) \setminus \{0,1\}$ which is not a solution of $x^2 - x + 1 = 0$. We can choose a suitable point $\mathcal{R}(x,1)$, since there exist at most two solutions of this equation and since $p^i > 5$ is proposed.
For getting such a point if p is odd, we have to regard, that x is neither an element of $\{0, 1, -1, 1/2, 2\}$ nor a solution of $x^2 - x + 1 = 0$; hence, $p^i > 7$ is sufficient.

(b) If at least two of the values of $(*)$ are equal, the four points are harmonic if the values of the cross-ratios are $\{-1, 1/2, 2\}$ or equianharmonic if $x = 1/(1-x)$ (or equivalently $x = (x-1)/x$) or both, which is called superharmonic by Hirschfeld and which occurs if and only if $p = 3$. In this case the stabilizer of B' is the symmetric group S_4 of order 24.
A harmonic quadruple where $p > 3$ is stabilized by the dihedral group D_4 of order 8. Equianharmonic quadruple exist exactly if $p^i \equiv 1 \mod 3$ and their stabilizer is the alternating group A_4 of order 12 ([6], p.121).

Let B' be a 4-subset of B, then we get the same groups as stabilizers as above.
Considering all the different possible stabilizers, we get, by proposition 2.1.1, the divisible designs of theorem 3.1.1, part (v).

□

Remark 3.1.1 A.G. Spera constructed divisible designs from a finite local K-algebra A, with $K = \mathrm{GF}(q)$ and J its Jacobson radical (with $|A| = q^n, |J| = q^j$, $n, j \in \mathbb{N}$). In the special case $K \cong A/J$, where A is a Laguerre algebra, he got a transversal $3-(q^j, q+1, 1)$-DD as in case (i) (cf. [10]).

3.2 - from a conic in a plane of a 3-dimensional projective space

In 1999, R.-H. Schulz and C. Cerroni constructed the following series of divisible designs [4].

Theorem 3.2.1 (Schulz, Cerroni) *Let $q = p^n$, where p is a prime number and let $n, i \in \mathbb{N}$ with $i|n$. If q is odd, there exists a $3-(q, p^i+1, 1)$ - DD with q^2+q points, having as a point- and block-transitive automorphism group $T\tilde{G}$ with $\tilde{G} \cong \mathrm{GO}(3,q)$ and T the translation group of $\mathrm{AG}(3,q)$.*

The proof can be found in [4]. In this paper we will show the idea of the construction only.

Consider a conic O in the ideal plane E' of the 3-dimensional affine space $\mathrm{AG}(3,q)$. Every point of O has a unique tangent, which determines exactly one parallel class of planes. Planes of the same parallel class all intersect E' in the appropriate tangent. In this way, one gets $q+1$ parallel classes of q planes each, which are the points of the constructed divisible designs and denotes this set by $\Pi(\Omega)$. Starting from $\mathrm{PGO}(3,q)$, which operates 3-transitively on O, Schulz and Cerroni constructed a 3-R-transitive R-permutation group (R denotes the parallelism relation) of $\Pi(\Omega)$. After choosing a base block in $\Pi(\Omega)$, they used Spera's proposition to construct the divisible designs.

3.3 Comparing both constructions

By remark 2.3.4, the points of the designs constructed from a Laguerre geometry in section 3.1 (cf. theorem 3.1.1) can be considered as the points of the $q+1$ lines of the cone $\mathcal{O}$ in a 3-dimensional projective space Ψ. Each line contains exactly one parallel class of points and all lines are concurrent in the point E. By dualising Ψ, we get the situation of the construction of Schulz and Cerroni in section 3.2. We have $q+1$ parallel classes of q planes each. All planes of these parallel classes intersect the plane E' which is dual to the point E. Therefore it is not surprising that we got isomorphic divisible designs from different constructions by choosing suitable base blocks.

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