REMARKS ON RICH SUBSPACES OF BANACH SPACES

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ABSTRACT. We investigate rich subspaces of L_1 and deduce an interpolation property of Sidon sets; for the definition see the introduction. We also present examples of rich separable subspaces of nonseparable Banach spaces and we study the Daugavet property of tensor products.

Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday

1. INTRODUCTION

In this paper we present some results concerning the notion of a rich subspace of a Banach space as introduced in [12]. In that paper (see also [18]), an operator $T: X \to Y$ is called narrow if for every $x, y \in S(X)$ (the unit sphere of X), $\varepsilon > 0$ and every slice S of the unit ball $B(X)$ of X containing y there is an element $v \in S$ such that $\|x + v\| > 2 - \varepsilon$ and $\|T(y - v)\| < \varepsilon$, and a subspace Z of X is called rich if the quotient map $q: X \to X/Z$ is narrow. We recall that a slice of the unit ball is a nonvoid set of the form $S = \{x \in B(X): \operatorname{Re} x^*(x) > \alpha\}$ for some functional $x^* \in X^*$. Thus, Z is a rich subspace if for every $x, y \in S(X)$, $\varepsilon > 0$ and every slice S of $B(X)$ containing y there is some $z \in X$ at distance $\leq \varepsilon$ from Z such that $y + z \in S$ and $\|x + y + z\| > 2 - \varepsilon$. Actually, we are not giving the original definition of a narrow operator but the equivalent reformulation from [12, Prop. 3.11].

These ideas build on previous work in [15] and [10]; however we point out that the above definition of richness is unrelated to Bourgain's in [4]. Narrow operators were used in [2] and [10] to extend Pełczyński's classical result that neither $C[0,1]$ nor $L_1[0,1]$ embed into spaces having unconditional bases.

The investigation of narrow operators is closely connected with the Daugavet property of a Banach space. A Banach space X has the Daugavet property whenever $\|\mathrm{Id} + T\| = 1 + \|T\|$ for every rank-1 operator $T: X \to X$; prime examples are $C(K)$ when K is perfect (i.e., has no isolated points),

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$L_1(\mu)$ and $L_\infty(\mu)$ when μ is nonatomic, the disc algebra, and spaces like $L_1[0,1]/V$ when V is reflexive. For future reference we mention the following characterisation of the Daugavet property [11]:

Lemma 1.1. *The following assertions are equivalent:*

(i) *X has the Daugavet property.*
(ii) *For every $x \in S(X)$, $\varepsilon > 0$ and every slice S of $B(X)$ there exists some $v \in S$ such that $\|x + v\| > 2 - \varepsilon$.*
(iii) *For all $x \in S(X)$ and $\varepsilon > 0$, $B(X) = \overline{\mathrm{co}}\{v \in B(X)\colon \|x+v\| > 2-\varepsilon\}$.*

Therefore, X has the Daugavet property if and only if 0 is a narrow operator on X or equivalently if and only if there exists at least one narrow operator on X. It is proved in [12] that then every weakly compact operator (indeed, every strong Radon-Nikodým operator) and every operator not fixing a copy of ℓ_1 is narrow (and hence satisfies $\|\mathrm{Id} + T\| = 1 + \|T\|$ when it maps X into X). Consequently, a subspace Z of a space with the Daugavet property is rich if X/Z or $(X/Z)^*$ has the RNP.

The general idea of these notions is that a narrow operator is sort of small and hence a rich subspace is large. In Section 2 of this paper we study rich subspaces of L_1. With reference to a quantity that is reminiscent of the Dixmier characteristic we show that a rich subspace is indeed large: a subspace with a bigger "characteristic" coincides with L_1. As an application we present an interpolation property of Sidon sets.

These results notwithstanding, Section 3 gives examples of rich subspaces that appear to be small, namely there are examples of nonseparable spaces and separable rich subspaces.

In Section 4 we study hereditary properties for the Daugavet property in tensor products. Although there are positive results for rich subspaces of $C(K)$, we present counterexamples in the general case.

2. Rich subspaces of L_1

Let $X \subset L_1 = L_1(\Omega, \Sigma, \lambda)$ be a closed subspace where λ is a probability measure. We define C_X to be the closure of $B(X)$ in L_1 with respect to the L_0-topology, the topology of convergence in measure. Note that for $f \in C_X$ there is a sequence (f_n) in $B(X)$ converging to f pointwise almost everywhere and almost uniformly. In this section, the symbol $\|f\|$ refers to the L_1-norm of a function.

In [12, Th. 6.1] narrow operators on the space L_1 were characterised as follows.

Theorem 2.1. *An operator $T\colon L_1 \to Y$ is narrow if and only if for every measurable set A and every $\delta, \varepsilon > 0$ there is a real-valued L_1-function f supported on A such that $\int f = 0$, $f \le 1$, the set $\{f = 1\}$ of those $t \in \Omega$ for which $f(t) = 1$ has measure $\lambda(\{f = 1\}) > \lambda(A) - \varepsilon$ and $\|Tf\| \le \delta$. In particular, a subspace $X \subset L_1$ is rich if and only if for every measurable set A and every $\delta, \varepsilon > 0$ there is a real-valued L_1-function f supported on A*

such that $\int f = 0$, $f \le 1$, $\lambda(\{f = 1\}) > \lambda(A) - \varepsilon$ *and the distance from* f *to* X *is* $\le \delta$.

Actually, in [12] only the case of real L_1-spaces was considered, but the proof extends to the complex case. Indeed, instead of the function v that is constructed in the first part of the proof of [12, Th. 6.1] one uses its real part and employs the fact that for real-valued L_1-functions v_1 and v_2 satisfying

$$1 - \delta < \int_\Omega |v_1|\, d\lambda \le \int_\Omega (v_1^2 + v_2^2)^{1/2}\, d\lambda \le 1$$

we have $\|v_2\| \le \sqrt{2\delta}$.

Proposition 2.2. *If* X *is rich, then* $\frac{1}{2}B(L_1) \subset C_X$.

Proof. Since C_X is L_1-closed, it is enough to show that $f_A := \chi_A/\lambda(A) \in 2C_X$ for every measurable set A. By Theorem 2.1 there is, given $\varepsilon > 0$, a real-valued function g_ε supported on A with $g_\varepsilon \le 1$ and $\int g_\varepsilon = 0$ such that $\{g_\varepsilon < 1\}$ has measure $\le \varepsilon$ and the distance of g_ε to X is $\le \varepsilon$. Clearly $g_\varepsilon/\lambda(A) \to f_A$ in measure as $\varepsilon \to 0$ and

$$\|g_\varepsilon\| = \|g_\varepsilon^+\| + \|g_\varepsilon^-\| = 2\|g_\varepsilon^+\| \le 2\lambda(A).$$

Therefore, there is a sequence (f_n) in X of norm ≤ 2 converging to f_A in measure. □

Proposition 2.3. *If* $\frac{1}{2}B(L_1) \subset C_Y$ *for all* 1*-codimensional subspaces* Y *of* X*, then* X *is rich.*

Proof. Again by Theorem 2.1, we have to produce functions g_ε as above on any given measurable set A. Therefore, we let $Y = \{f \in X\colon \int_A f = 0\}$. By assumption, there is a sequence (f_n) in Y such that $\|f_n\| \le 2\lambda(A)$ and $f_n \to \chi_A$ in measure.

We shall argue that $\|\operatorname{Im} f_n\| \to 0$. Let $\eta > 0$. If n is large enough, the set $B_n := \{|f_n - \chi_A| \ge \eta\}$ has measure $\le \eta$. For those n,

$$0 = \int_A \operatorname{Re} f_n = \int_{A\setminus B_n} \operatorname{Re} f_n + \int_{A\cap B_n} \operatorname{Re} f_n$$

implies that

$$\int_{A\cap B_n} |\operatorname{Re} f_n| \ge \left| \int_{A\cap B_n} \operatorname{Re} f_n \right| = \left| \int_{A\setminus B_n} \operatorname{Re} f_n \right| \ge \lambda(A \setminus B_n)(1-\eta)$$

and

$$\|\operatorname{Re} f_n|_A\| \ge \lambda(A\setminus B_n)(1-\eta) + \int_{A\cap B_n} |\operatorname{Re} f_n| \ge 2(\lambda(A) - \eta)(1-\eta).$$

Hence,

$$2(\lambda(A) - \eta)(1-\eta) \le \|\operatorname{Re} f_n|_A\| \le \|f_n|_A\| \le \|f_n\| \le 2\lambda(A),$$

and it follows for one thing that $\|\mathrm{Im}\, f_n|_A\|$ is small provided η is small enough (cf. the remarks after Theorem 2.1) and moreover that

$$\|f_n|_{[0,1]\setminus A}\| \le 2\eta + 2\eta\lambda(A).$$

Consequently, $\|\mathrm{Im}\, f_n\| \to 0$ as $n \to \infty$.

Now let $\delta = \varepsilon/9$ and choose n so large that the set $B := \{|\mathrm{Re}\, f_n - \chi_A| \ge \delta\}$ has measure $\le \delta$ and $\|\mathrm{Im}\, f_n\| \le \delta$. Then there exists a real-valued function h such that $h = 0$ on $[0,1] \setminus (A \cup B)$, $h = 1$ on $A \setminus B$, $\int_A h = 0$ and $\|h - \mathrm{Re}\, f_n\| \le 2\delta$. Now

$$\begin{aligned} \|h|_A\| &= 2\|h^+|_A\| \ge 2(\lambda(A) - \delta) \\ \|h\| &\le \|\mathrm{Re}\, f_n\| + 2\delta \le 2(\lambda(A) + \delta), \end{aligned}$$

so

$$\|h|_{[0,1]\setminus A}\| \le 4\delta.$$

Furthermore,

$$\begin{aligned} \|h^+|_A\| &= \|h^+|_{A\cap B}\| + \|h^+|_{A\setminus B}\| \ge \|h^+|_{A\cap B}\| + \lambda(A) - \delta, \\ 2\|h^+|_A\| &= \|h|_A\| \le 2(\lambda(A) + \delta), \end{aligned}$$

so

$$\|h^+|_{A\cap B}\| \le 2\delta,$$

and it follows that there is a function $g = g_\varepsilon$ such that $g = 0$ on $[0,1] \setminus A$, $g = 1$ on $A \setminus B$, $\int g = 0$, $g \le 1$ and $\|g - h\| \le 4\delta$. Then

$$\mathrm{dist}(g, X) \le \|g - f_n\| \le \|g - h\| + \|h - \mathrm{Re}\, f_n\| + \|\mathrm{Im}\, f_n\| \le 9\delta = \varepsilon,$$

as requested. ☐

Since a 1-codimensional subspace of a rich subspace is rich [11, Th. 5.12], Proposition 2.2 shows that Proposition 2.3 can actually be formulated as an equivalence. This is not so for Proposition 2.2: the space constructed in Theorem 6.3 of [12] is not rich, yet it satisfies $\frac{1}{2}B(L_1) \subset C_X$.

We sum this up in a theorem.

Theorem 2.4. *X is a rich subspace of L_1 if and only if $\frac{1}{2}B(L_1) \subset C_Y$ for all 1-codimensional subspaces Y of X.*

The next proposition shows that the factor $\frac{1}{2}$ is optimal.

Proposition 2.5. *If, for some $r > \frac{1}{2}$, $rB(L_1) \subset C_X$, then $X = L_1$.*

Proof. Suppose $h \in L_\infty$, $\|h\|_\infty = 1$, and let $Y = \{f \in L_1\colon \int fh = 0\}$. Assume that $B(L_1) \subset sC_Y$; we shall argue that $s \ge 2$. This will prove the proposition since every proper closed subspace is contained in a closed hyperplane.

Assume without loss of generality that h takes the (essential) value 1. Let $\varepsilon > 0$, and put $A = \{|h - 1| < \varepsilon/2\}$; then A has positive measure. There is a sequence (f_n) converging to χ_A in measure such that $\|f_n\| \le s\,\lambda(A)$ and $\int f_n h = 0$ for all n. Since $f_n h \to \chi_A h$ in measure as well, there is, if n is a sufficiently large index, a subset $A_n \subset A$ of measure $\geq (1-\varepsilon)\lambda(A)$ such that $|f_n h - 1| < \varepsilon$ on A_n. For such an n,

$$\begin{aligned}\left|\int_{A_n} f_n h\right| &= \left|\lambda(A_n) - \int_{A_n} (1 - f_n h)\right| \\ &\geq \lambda(A_n) - \int_{A_n} |1 - f_n h| \geq (1-\varepsilon)\lambda(A_n),\end{aligned}$$

and therefore

$$\int_{A_n} |f_n h| \geq (1-\varepsilon)\lambda(A_n)$$

and, if B_n denotes the complement of A_n,

$$\int_{B_n} |f_n h| \geq \left|\int_{B_n} f_n h\right| = \left|\int_{A_n} f_n h\right| \geq (1-\varepsilon)\lambda(A_n)$$

so that

$$s\,\lambda(A) \geq \|f_n\| \geq \|f_n h\| \geq 2(1-\varepsilon)^2\lambda(A).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $s \geq 2$. □

Thus, the rich subspaces appear to be the next best thing in terms of size of a subspace after L_1 itself. At the other end of the spectrum are the nicely placed subspaces, defined by the condition that $B(X)$ is L_0-closed. Recall that X is nicely placed if and only if X is an L-summand in its bidual, i.e., $X^{**} = X \oplus_1 X_s$ (ℓ_1-direct sum) for some closed subspace X_s of X^{**} [8, Th. IV.3.5].

We next formulate a lemma that is probably known. We now assume in addition that Ω is a compact space and that λ is regular.

Lemma 2.6. *If (f_n) is a sequence in L_1 tending to 0 almost everywhere and tending to the measure μ for the weak* topology $\sigma(M(\Omega), C(\Omega))$, then μ is singular.*

Proof. Let $\varepsilon > 0$ and $A_n = \bigcap_{k\geq n}\{|f_k| \leq \varepsilon\}$. Pick compact sets $K_n \subset A_n$ and open sets $U_n \supset A_n$ with $\lambda(U_n \setminus K_n) \leq 2^{-n}$. Let $\varphi_n \colon \Omega \to [0,1]$ be a continuous function that vanishes off U_n and takes the value 1 on K_n. Then $\varphi_n \to 1$ almost everywhere.

Now, for fixed n, $\varphi_n f_k \to \varphi_n \mu$ weak* as $k \to \infty$. The set $W := \{h \in L_1\colon |h| \leq \varepsilon\}$ is a weakly compact subset of L_1, and $\varphi_n f_k \in W$ for $k \geq n$. Thus $\varphi_n \mu \in W$ and in particular $\varphi_n \mu \in L_1$.

Let $\mu = f\lambda + \nu$ be the Lebesgue decomposition of μ into an absolutely continuous and a singular component. Then $\varphi_n \nu = \varphi_n \mu - \varphi_n f\lambda$, and the left hand side is singular and the right hand side is absolutely continuous. Hence both sides vanish, and it follows that $\varphi_n f\lambda = \varphi_n \mu$. This implies that $|\varphi_n f| \leq \varepsilon$ for all n and, consequently, $|f| \leq \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, we have $f = 0$, and $\mu = \nu$ is singular. □

We now look at the translation invariant case, and we consider $L_1(\mathbb{T})$ (or $L_1(G)$ for a compact metrisable abelian group). As usual, for $\Lambda \subset \mathbb{Z}$ the space $L_{1,\Lambda}$ (resp. M_Λ) consists of those L_1-functions (resp. measures) whose Fourier transforms vanish off Λ.

Proposition 2.7. *For a subset $\Lambda \subset \mathbb{Z}$, the following are equivalent:*

(i) $\frac{1}{2}B(L_1) \subset C_{L_{1,\Lambda}}$.

(ii) *For every measure μ on $\mathbb{T}$ there is a singular measure ν with $\|\nu\| \le \|\mu\|$ and $\widehat{\nu}(\gamma) = \widehat{\mu}(\gamma)$ for all $\gamma \notin \Lambda$.*

Proof. (i) $\Rightarrow$ (ii): Let $\mu = f\lambda + \mu_s$ be the Lebesgue decomposition of μ, and let (f_n) be a sequence in $L_{1,\Lambda}$ such that $f_n \to f$ almost everywhere and $\|f_n\| \le 2\|f\|$. By passing to a subsequence we assume that (f_n) weak* converges to some measure μ'; note that $\|\mu'\| \le 2\|f\|$. Put $\mu'_s = \mu' - f\lambda$; Lemma 2.6 implies that $\mu' = f\lambda + \mu'_s$ is the Lebesgue decomposition of that measure. Since $f_n \in L_{1,\Lambda}$, we have that $\mu' \in M_\Lambda$. Therefore, the singular measure $\nu = \mu_s - \mu'_s$ has the same Fourier coefficients outside Λ as μ, and

$$\|\nu\| \le \|\mu_s\| + \|\mu'_s\| \le \|\mu_s\| + \|f\| = \|\mu\|,$$

for

$$2\|f\| \ge \|\mu'\| = \|f\| + \|\mu'_s\|.$$

(ii) $\Rightarrow$ (i): Let $f \in L_1$. Pick a singular measure ν as in (ii) for the absolutely continuous measure $\mu = f\lambda$; then $\alpha := f\lambda - \nu \in M_\Lambda$ and $\|\alpha\| \le 2\|f\|$. Hence there is a sequence (f_n) in $L_{1,\Lambda}$ such that $f_n \to \alpha$ weak* and $\|f_n\| \le 2\|f\|$ for all n. By Komlos's theorem there is a sequence (g_n) of convex combinations of the f_n converging almost everywhere. Since still $g_n \to \alpha$ weak*, it follows from Lemma 2.6 that $g_n \to f$ almost everywhere. To complete the proof note that $\|g_n\| \le 2\|f\|$ and $g_n \in L_{1,\Lambda}$. $\square$

We apply these ideas to Sidon sets, i.e., sets $\Lambda' \subset \mathbb{Z}$ such that all functions in $C_{\Lambda'}$ have absolutely sup-norm convergent Fourier series. (See [13] for recent results on this notion.) If Λ is the complement of a Sidon set, then $L_1/L_{1,\Lambda}$ is isomorphic to c_0 or finite-dimensional [16, p. 121]. Hence $L_{1,\Lambda}$ is rich by [12, Prop. 5.3]; therefore the interpolating property (ii) above is true for a co-Sidon set Λ. Thus, the following corollary holds.

Corollary 2.8. *If $\Lambda' \subset \mathbb{Z}$ is a Sidon set and μ is a measure on $\mathbb{T}$, then there is a singular measure ν with $\|\nu\| \le \|\mu\|$ and $\widehat{\nu}(\gamma) = \widehat{\mu}(\gamma)$ for all $\gamma \in \Lambda'$.*

To show that there are also non-Sidon sets sharing this property we observe a simple lemma.

Lemma 2.9. *If Z is a rich subspace of X, then $L_1(Z)$ is a rich subspace of the Bochner space $L_1(X)$.*

Proof. It is enough to check the definition of narrowness of the quotient map on vector-valued step functions. Thus the assertion of the lemma is reduced to the assertion that $Z \oplus_1 \cdots \oplus_1 Z$ is a rich subspace of $X \oplus_1 \cdots \oplus_1 X$; but this has been proved in [3]. $\square$

Now if $\Lambda \subset \mathbb{Z}$ is a co-Sidon set, then $L_1(L_{1,\Lambda}) \cong L_{1,\mathbb{Z}\times\Lambda}(\mathbb{T}^2)$ is a rich subspace of $L_1(L_1) \cong L_1(\mathbb{T}^2)$, and $\Lambda' = \mathbb{Z} \times (\mathbb{Z}\backslash\Lambda)$ is a non-Sidon set with reference to the group $\mathbb{T}^2$ for which Corollary 2.8 is valid.

3. Some examples of small but rich subspaces

In this section we provide examples of nonseparable Banach spaces and separable rich subspaces.

First we give a handy reformulation of richness. We let

$$D(x,y,\varepsilon) = \{z \in X\colon \|x+y+z\| > 2-\varepsilon,\ \|y+z\| < 1+\varepsilon\}$$

for $x, y \in S(X)$.

Lemma 3.1. *The following assertions are equivalent for a Banach space X.*

(i) *Z is a rich subspace of X.*

(ii) *For every $x, y \in S(X)$ and every $\varepsilon > 0$,*

$$y \in \overline{\mathrm{co}}\big(y + \big(D(x,y,\varepsilon) \cap Z\big)\big).$$

(iii) *For every $x, y \in S(X)$ and every $\varepsilon > 0$,*

$$0 \in \overline{\mathrm{co}}\big(D(x,y,\varepsilon) \cap Z\big).$$

Proof. (i) $\Leftrightarrow$ (ii) is a consequence of the Hahn-Banach theorem, and (ii) $\Leftrightarrow$ (iii) is obvious. $\square$

For $Z = X$, (ii) boils down to condition (iii) of Lemma 1.1.

In the examples we are going to present Z will be a space $C(K,E)$ embedded in a suitable space X. The type of space we have in mind will be defined next.

Definition 3.2. Let E be a Banach space and X be a sup-normed space of bounded E-valued functions on a compact space K. The space X is said to be a $C(K,E)$-superspace if it contains $C(K,E)$ and for every $f \in X$, every $\varepsilon > 0$ and every open subset $U \subset K$ there exists an element $e \in E$, $\|e\| > (1-\varepsilon)\sup_U \|f(t)\|$, and a nonvoid open subset $V \subset U$ such that $\|e - f(\tau)\| < \varepsilon$ for every $\tau \in V$.

Basically, X is a $C(K,E)$-superspace if every element of X is large and almost constant on suitable open sets.

Here are some examples of this notion.

Proposition 3.3.

(a) *$D[0,1]$, the space of bounded functions on $[0,1]$ that are right-continuous and have left limits everywhere and are continuous at $t=1$, is a $C[0,1]$-superspace.*

(b) *Let K be a compact Hausdorff space and E be a Banach space. Then $C_w(K,E)$, the space of weakly continuous functions from K into E, is a $C(K,E)$-superspace.*

Proof. (a) $D[0,1]$ is the uniform closure of the span of the step functions $\chi_{[a,b)}$, $0 \le a < b < 1$, and $\chi_{[a,1]}$, $0 \le a < 1$; hence the result.

(b) Fix f, U and ε as in Definition 3.2; without loss of generality we assume that $\sup_U \|f(t)\| = 1$. Consider the open set $U_0 = \{t \in U\colon \|f(t)\| > 1-\varepsilon\}$. Now $f(U_0)$ is relatively weakly compact since f is weakly continuous; hence it is dentable [1, p. 110]. Therefore there exists a halfspace $H = \{x \in E\colon x^*(x) > \alpha\}$ such that $f(U_0) \cap H$ is nonvoid and has diameter $< \varepsilon$. Consequently, $V := f^{-1}(H) \cap U_0$ is an open subset of U for which $\|f(\tau_1) - f(\tau_2)\| < \varepsilon$ for all $\tau_1, \tau_2 \in V$. This shows that $C_w(K,E)$ is a $C(K,E)$-superspace. $\square$

The following theorem explains the relevance of these ideas.

Theorem 3.4. *If X is a $C(K,E)$-superspace and K is perfect, then $C(K,E)$ is rich in X; in particular, X has the Daugavet property.*

Proof. We wish to verify condition (iii) of Lemma 3.1. Let $f, g \in S(X)$ and $\varepsilon > 0$. We first find an open set V and an element $e \in E$, $\|e\| > 1-\varepsilon/4$, such that $\|e - f(\tau)\| < \varepsilon/4$ on V. Given $N \in \mathbb{N}$, find open nonvoid pairwise disjoint subsets $V_1, \dots, V_N$ of V. Applying the definition again, we obtain elements $e_j \in E$ and open subsets $W_j \subset V_j$ such that $\|e_j\| > (1-\varepsilon/4)\sup_{V_j} \|g(t)\|$ and $\|e_j - g(\tau)\| < \varepsilon/4$ on W_j. Let $x_j = e - e_j$, let $\varphi_j \in C(K)$ be a positive function supported on W_j of norm 1 and let $h_j = \varphi_j \otimes x_j$. Now if $t_j \in W_j$ is selected to satisfy $\varphi_j(t_j) = 1$, then

$$\|f + g + h_j\| \ge \|(f + g + h_j)(t_j)\| > \|e + e_j + x_j\| - \varepsilon/2 > 2 - \varepsilon$$

and

$$\|g + h_j\| < 1 + \varepsilon$$

since $\|g(t) + h_j(t)\| \le 1$ for $t \notin W_j$, and for $t \in W_j$

$$\|g(t) + h_j(t)\| \le \|e_j + \varphi_j(t)x_j\| + \varepsilon/4 \le (1 - \varphi_j(t))\|e_j\| + \varphi_j(t)\|e\| + \varepsilon/4.$$

This shows that $h_j \in D(f, g, \varepsilon) \cap C(K,E)$. But the supports of the h_j are pairwise disjoint, hence $\|1/N \sum_{j=1}^N h_j\| \le 2/N \to 0$.

That X has the Daugavet property follows from the fact that there is a narrow operator on X. $\square$

Corollary 3.5.

(a) *$C[0,1]$ is a separable rich subspace of the nonseparable space $D[0,1]$.*

(b) *If K is perfect, then $C(K,E)$ is a rich subspace of $C_w(K,E)$. In particular, $C([0,1],\ell_p)$ is a separable rich subspace of the nonseparable space $C_w([0,1],\ell_p)$ if $1 < p < \infty$.*

Let us remark that there exist nonseparable spaces with the Daugavet property with only nonseparable rich subspaces. Indeed, an ℓ_∞-sum of uncountably many spaces with the Daugavet property is an example of this phenomenon. To see this we need the result from [3] that whenever T is a narrow operator on $X_1 \oplus_\infty X_2$, then the restriction of T to X_1 is narrow too, and in particular it is not bounded from below. Now let X_i, $i \in I$, be Banach spaces with the Daugavet property and let X be their ℓ_∞-sum. If Z is a rich subspace of X, then by the result quoted above there exist elements $x_i \in S(X_i)$ and $z_i \in Z$ with $\|x_i - z_i\| \le 1/4$; hence $\|z_i - z_j\| \ge 1/2$ for $i \ne j$. If I is uncountable, this implies that Z is nonseparable.

4. The Daugavet property and tensor products

One may consider the space $C(K,E)$ as the injective tensor product of $C(K)$ and E; see for instance [6, Chap. VIII] or [17, Chap. 3] for these matters. It is known that $C(K,E)$ has the Daugavet property whenever $C(K)$ has, regardless of E ([9] or [11]), and it is likewise true that $C(K,E)$ has the Daugavet property whenever E has, regardless of K [14]. This raises the natural question whether the injective tensor product of two spaces has the Daugavet property if at least one factor has.

We first give a positive answer for the class of rich subspaces of $C(K)$; for example, a uniform algebra is a rich subspace of $C(K)$ if K denotes its Silov boundary and is perfect.

Proposition 4.1. *If X is a rich subspace of some $C(K)$-space, then $X \widehat{\otimes}_\varepsilon E$, the completed injective tensor product of X and E, is a rich subspace of $C(K)\widehat{\otimes}_\varepsilon E$ for every Banach space E; in particular, it has the Daugavet property.*

Proof. We will consider $X \widehat{\otimes}_\varepsilon E$ as a subspace of $C(K,E)$. In order to verify (iii) of Lemma 3.1 let $f, g \in S(C(K,E))$ and $\varepsilon > 0$ be given. Further, let $\eta > 0$ be given. We wish to construct functions $h_1, \dots, h_n \in D(f,g,\varepsilon) \cap X \widehat{\otimes}_\varepsilon E$ such that $\|\frac{1}{n}\sum_{j=1}^n h_j\| \le 2\eta$.

There is no loss in assuming that $\eta \le \varepsilon$. Consider $U = \{t\colon \|f(t)\| > 1 - \eta/2\}$. By reducing U if necessary we may also assume that $\|g(t) - g(t')\| < \eta$ for $t, t' \in U$. Fix $n \ge 2/\eta$ and pick n pairwise disjoint open nonvoid subsets $U_1, \dots, U_n$ of U; this is possible since K must be perfect, for $C(K)$ carries a narrow operator, viz. the quotient map $q\colon C(K) \to C(K)/X$. By applying [12, Th. 3.7] to q we infer that there exists, for each j, a function $\psi_j \in X$ with $\psi_j \ge 0$, $\|\psi_j\| = 1$ and $\psi_j < \eta/2$ off U_j. Choose $t_j \in U_j$ with $\psi_j(t_j) = 1$. We define

$$h_j = \psi_j \otimes (f(t_j) - g(t_j)) \in X \widehat{\otimes}_\varepsilon E$$

and claim that $h_j \in D(f,g,\eta) \subset D(f,g,\varepsilon)$. In fact,

$$\|f + g + h_j\| \ge \|f(t_j) + g(t_j) + h_j(t_j)\| = 2\|f(t_j)\| > 2 - \eta.$$

Also, $\|g + h_j\| < 1 + \eta$, for if $t \in U_j$, then

$$\begin{aligned}\|g(t) + h_j(t)\| &\le \|g(t_j) + h_j(t)\| + \|g(t) - g(t_j)\| \\ &< \|(1 - \psi_j(t))g(t_j) + \psi_j(t) f(t_j)\| + \eta \\ &\le 1 + \eta,\end{aligned}$$

and for $t \notin U_j$ we clearly have $\|g(t) + h_j(t)\| < 1 + \eta$.

It is left to estimate $\|\frac{1}{n}\sum_{j=1}^n h_j\|$. If t does not belong to any of the U_j, we have

$$\left\|\frac{1}{n}\sum_{j=1}^n h_j(t)\right\| \le \eta,$$

and if $t \in U_i$, we have

$$\left\|\frac{1}{n}\sum_{j=1}^n h_j(t)\right\| \le \frac{n-1}{n}\eta + \frac{1}{n}\|h_i(t)\| \le \eta + \frac{2}{n} \le 2\eta$$

by our choice of n. □

In general, however, the above question has a negative answer.

Theorem 4.2. *There exists a two-dimensional complex Banach space E such that $L_1^{\mathbb{C}}[0,1]\widehat{\otimes}_\varepsilon E$ fails the Daugavet property, where $L_1^{\mathbb{C}}[0,1]$ denotes the space of complex-valued L_1-functions.*

Proof. Consider the subspace E of complex ℓ_∞^6 spanned by the vectors $x_1 = (1,1,1,1,1,0)$ and $x_2 = (0,\frac{1}{2},-\frac{1}{2},\frac{i}{2},-\frac{i}{2},1)$. The injective tensor product of E and $L_1^{\mathbb{C}}[0,1]$ can be identified with the space of 6-tuples of functions $f = (f_1,\dots,f_6)$ of the form $g_1 \otimes x_1 + g_2 \otimes x_2$, $g_1, g_2 \in L_1^{\mathbb{C}}[0,1]$, with the norm $\|f\| = \max_{k=1,\dots,6}\|f_k\|_1$. To show that this space does not have the Daugavet property, consider the slice

$$S_\varepsilon = \left\{ f = (f_1,\dots,f_6) \in L_1^{\mathbb{C}}[0,1]\otimes E\colon \operatorname{Re}\int_0^1 f_1(t)\,dt > 1-\varepsilon,\ \|f\| \le 1 \right\}.$$

Every $f = g_1 \otimes x_1 + g_2 \otimes x_2 \in S_\varepsilon$ satisfies the conditions

$$\|g_1\| > 1-\varepsilon, \quad \max\{\|g_1 \pm \tfrac{1}{2}g_2\|, \|g_1 \pm \tfrac{i}{2}g_2\|\} \le 1.$$

Now the complex space L_1 is complex uniformly convex [7]. Therefore, there exists a function $\delta(\varepsilon)$, which tends to 0 when ε tends to 0, such that $\|g_2\| < \delta(\varepsilon)$ for every $f = g_1 \otimes x_1 + g_2 \otimes x_2 \in S_\varepsilon$. This implies that for every $f \in S_\varepsilon$

$$\|1 \otimes x_2 + f\| \le \frac{3}{2} + \delta(\varepsilon).$$

So if ε is small enough, there is no $f \in S_\varepsilon$ with $\|1 \otimes x_2 + f\| > 2-\varepsilon$. By Lemma 1.1, this proves that this injective tensor product does not have the Daugavet property. □

For the projective norm it is known that $L_1(\mu)\widehat{\otimes}_\pi E = L_1(\mu, E)$ has the Daugavet property regardless of E whenever μ has no atoms [11]. Again, there is a counterexample in the general case.

Corollary 4.3. *There exists a two-dimensional complex Banach space F such that $L_\infty^{\mathbb{C}}[0,1]\widehat{\otimes}_\pi F$ fails the Daugavet property, where $L_\infty^{\mathbb{C}}[0,1]$ denotes the space of complex-valued L_∞-functions.*

Proof. Let E be the two-dimensional space from Theorem 4.2; note that $(L_1^{\mathbb{C}}\widehat{\otimes}_\varepsilon E)^* = L_\infty^{\mathbb{C}}\widehat{\otimes}_\pi E^*$. Since the Daugavet property passes from a dual space to its predual, $F := E^*$ is the desired example. □

5. Questions

We finally mention two questions that were raised by A. Pełczyński which we haven't been able to solve.

(1) Is there a rich subspace of L_1 with the Schur property? Is there a Schur space with the Daugavet property? A candidate for such an example could be the subspace $X \subset L_1$ constructed by Bourgain and Rosenthal in [5]. For this space, the results in [5] imply that $\|\mathrm{Id} - P\| = 2$ for every rank-1 projection P of norm 1.

(2) If X is a subspace of L_1 with the RNP, does L_1/X have the Daugavet property? The answer is positive for reflexive spaces [11], for H^1 [19] and a certain space constructed by Talagrand in his solution of the three-space problem for L_1 [11].

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