Generalized Albanese morphisms

Georg Hein

April 8, 2003

Abstract

We define generalizations of the Albanese variety for a projective variety X. The generalized Albanese morphisms $X \overset{\mathrm{alb}_r}{\dashrightarrow} \mathrm{Alb}_r(X)$ contract those curves C in X for which the induced morphism $\mathrm{Hom}(\pi_1(X), \mathrm{U}(r)) \to \mathrm{Hom}(\pi_1(C), \mathrm{U}(r))$ has a finite image. Thus, they may be interpreted as a $\mathrm{U}(r)$-version of the Shafarevich morphism.

1 Introduction

The Shafarevich conjecture states that the universal cover $\tilde{X}$ of a projective variety X is holomorphically convex. This means that there exists a proper surjection $\tilde{X} \to Y$ to a Stein space. This implies (see [13]) the existence of a morphism $\mathrm{sh}_X : X \to \mathrm{Sh}(X)$ with connected fibers with the property that a curve C in X is contracted by sh_X, iff the image of $\pi_1(C)$ in $\pi_1(X)$ is finite. The Shafarevich variety is the quotient of the Stein space Y by the fundamental group $\pi_1(X)$ of X.
The Shafarevich conjecture is very hard to tackle in terms of algebraic geometry because the holomorphic map $\tilde{X} \to X$ is not a regular morphism of algebraic varieties unless the fundamental group of X is finite. It is, however, possible to deal with the problem of finding the morphism sh_X. A first version of such a morphism was given by Kollár in [13]. For Kähler manifolds, Campana constructs [4] a generic version of the Shafarevich morphism. Katzarkov (see [11] and [12]) describes a representation description of sh_X, for one representation $\pi_1(X) \to \mathrm{GL}(r)$. We offer a different approach to such a description for all representations by considering generalized Albanese varieties.
Two constructions for the Albanese variety exist. One construction (2.1) uses global holomorphic one forms while another (2.2) defines the Albanese variety as the dual of the Picard torus. The intermediate Jacobians are a generalization of the first construction. Here we try to show how to generalize the second construction. In the sense of Kollár [13], where the Albanese variety can be see as the abelian or $\mathbb{C}^*$-version of the Shafarevich morphism, we intend to give a $\mathrm{U}(r)$-version of the Shafarevich morphism that is a map $\mathrm{alb}_r(X) : X \to \mathrm{Alb}_r(X)$ having connected fibers, such that an irreducible curve C in X is contracted under $\mathrm{alb}_r(X)$, iff the image of the composition $\mathrm{Hom}(\pi_1(X), \mathrm{U}(r)) \to \mathrm{Hom}(\pi_1(C), \mathrm{U}(r))$ has finite image up to conjugation. If the image of $\pi_1(C)$ in $\pi_1(X)$ is finite, then C is contracted by the generalized Albanese morphisms. Therefore the generalized Albanese morphisms would factor over the Shafarevich morphism.
In section 2 we review the constructions and some properties of the Albanese variety. The construction of the generalized Albanese morphism is first presented under two simplifying assumptions in 3.1, and then in a full generalization in section 3.2. The main result is the

existence of a rational map $X \stackrel{\mathrm{alb}_r}{\dashrightarrow} \mathrm{Alb}_r(X)$ with the above property. The result of the generalized Theta divisor (3.11) will be proved in the fourth section. In the fifth section we go over the construction for the case of a projective algebraic surface X. Here we are able to deduce the existence of regular morphisms $X \xrightarrow{\mathrm{alb}_r} \mathrm{Alb}_r(X)$ for all integers $r \geq 1$. Lemma 5.6 might independently be of interest. In the last section we give an example of a surface with a classical Albanese variety of dimension one, and generalized Albanese varieties of dimension two. Thus, this example shows that the generalized Albanese morphism may "reveal" more than the classical Albanese morphism does.
Since we need the restriction of (semi)stable vector bundles to curves, we have to use the concept of Mumford or μ-stability for vector bundles.

2 Two constructions for the Albanese variety

2.1 The classical construction for the Albanese variety. Here we assume that X is a connected Kähler manifold. We define the Albanese variety $\mathrm{Alb}(X)$ to be the quotient

$$\mathrm{Alb}(X) := H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z})\,.$$

If we choose a point $x_0 \in X(\mathbb{C})$, then we can define the Albanese morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ by $x \mapsto \int_{\gamma_x}$ where γ_x is a path connecting x_0 with x.

2.2 The $\mathrm{Pic}^0(\mathrm{Pic}^0)$-description of $\mathrm{Alb}(X)$. Let X be a smooth variety over an algebraically closed field k. We consider the Picard torus $\mathrm{Pic}^0(X)$, i.e., the component of $\mathrm{Pic}(X)$ containing $\mathcal{O}_X$. Furthermore, we consider a Poincaré bundle L on $X \times \mathrm{Pic}^0(X)$. This bundle is not unique. To normalize it we choose a point $x_0 \in X(k)$. If we require that $L|_{\{x_0\} \times \mathrm{Pic}^0(X)} \cong \mathcal{O}_{\mathrm{Pic}^0(X)}$, then the Poincaré bundle L is uniquely determined. If we consider L as a family of line bundles on $\mathrm{Pic}^0(X)$ parametrized by X, then we obtain a morphism from X to the Picard torus of $\mathrm{Pic}^0(X)$:

$$\mathrm{alb}_X : X \to \mathrm{Pic}^0(\mathrm{Pic}^0(X)) =: \mathrm{Alb}(X)\,.$$

2.3 Both constructions coincide for smooth varieties over $\mathrm{Spec}(\mathbb{C})$. The use of the same notations in the above constructions is justified, because both coincide for a smooth projective variety over $\mathrm{Spec}(\mathbb{C})$. This follows from the universal property of the Albanese variety and the duality between the Albanese variety and the Picard torus (cf. [9]). The following proposition describes the fibers of the Albanese morphism. Since it uses both descriptions (2.1 and 2.2), we have to assume that $X/\mathrm{Spec}(\mathbb{C})$ is a smooth projective variety.

2.4 Proposition. (Description of the fibers of the Albanese morphisms $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$, for a projective complex manifold X (cf. II.6 in [9]).)
If $\iota : Z \to X$ is a connected cycle in X, then the following conditions are equivalent:
(i) Z is contained in a fiber of the morphism alb_X;

(ii) The image of $\iota_* : H_1(Z, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is finite;

(iii) The pull back morphism $\iota^* : H^0(X, \Omega^1_X) \to H^0(Z, \Omega^1_Z)$ is trivial;

(iv) Let $\rho : \pi_1(X) \to \mathbb{C}^*$ be a representation of the fundamental group. Then the restriction $\rho|_{\pi_1(Z)}$ has finite a image;

(v) If L is a line bundle on $X \times S$, then the pull back $\iota^* L$ on $Z \times S$ is of the form $L_1 \boxtimes L_2$, for any noetherian scheme S.

3 Construction of generalized Albanese varieties

3.1 Sketch of the construction

3.1 Let (X, H) be a polarized projective variety over an algebraically closed field k. Furthermore, we fix an integer $r \geq 1$ and classes $c_i \in H^{2i}(X, \mathbb{Z})$, for $i = 1, \ldots, r$. In 3.4 we present a construction of the generalized Albanese variety. However, this presentation is simplified by the following two assumptions (3.2 and 3.3). Unfortunately, these assumptions do not hold unless $r = 1$ (see 3.5). The main part of this section explains how to avoid these two assumptions.

3.2 Assumption. There exists a fine moduli space $\mathcal{M}$ for semistable vector bundles E (more precisely coherent sheaves) on X of rank r with Chern classes $c_i(E) = c_i$, for $i = 1, \ldots, r$.

Under this assumption, we deduce the existence of a universal bundle $\mathcal{E}$ on the product $X \times \mathcal{M}$. Just as in the case of the Poincaré bundle on the Picard torus, such a universal object $\mathcal{E}$ is not unique. However, suppose we have fixed one coherent universal sheaf $\mathcal{E}$.

3.3 Assumption. The universal sheaf $\mathcal{E}$ is flat with respect to the projection $X \times \mathcal{M} \to X$. Moreover, there exists a polarization $H_{\mathcal{M}}$ on $\mathcal{M}$ such that for each point $x \in X(k)$ the coherent sheaf $\mathcal{E}_x := \mathcal{E}|_{\{x\} \times \mathcal{M}}$ is semistable with Chern classes c'_i.

3.4 The construction of the generalized Albanese variety. Under these assumptions we obtain a morphism ψ from X to the (coarse) moduli space $\mathcal{M}'$ of semistable sheaves on $\mathcal{M}$ of rank r with given Chern classes c'_i. If $r = 1$, then this is exactly the $\mathrm{Pic}^0(\mathrm{Pic}^0)$ construction 2.2 for the Albanese variety. This explains why we call it the generalized Albanese variety. However, we prefer to move on to the Stein factorization

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}_r"] \arrow[dr, "\psi"'] & \mathrm{Alb}_r(X) \arrow[d] \\ & \mathcal{M}' \end{tikzcd}$$

of ψ. This way the generalized Albanese morphism $\mathrm{alb}_r : X \to \mathrm{Alb}_r(X)$ can be characterized by its fibers. As in the case of line bundles we would like to have (see 2.4 (v)) all the sheaves parametrized by $\mathcal{M}$ are pull backs of sheaves defined on $\mathrm{Alb}_r(X)$. This wishful description tells us that to study the rank r vector bundles on X (with given Chern classes c_i) it is enough to study them on $\mathrm{Alb}_r(X)$. Thus, the higher the fiber dimension of alb_r, the more useful this morphism becomes.

3.5 How to choose the Chern classes c_i? The generalized Albanese morphism is particularly interesting if $\dim(X) > \dim(\mathrm{Alb}_r(X))$. This can only be the case if all the Chern numbers $\int_X \prod_{i=1}^r c_i^{a_i}$ vanish. There is one canonical choice for the Chern classes satisfying this condition, namely to set $c_i = 0$, for $i = 1, \ldots, r$. There might be other interesting choices for the c_i which would require writing $\mathrm{Alb}_{r,c_1,\ldots,c_r}(X)$. We content ourselves with the case where all Chern classes are numerically equal to zero.
This choice implies that the moduli space is not empty, because the trivial bundle $\mathcal{O}_X^{\oplus r}$ has trivial Chern classes. Furthermore, its restriction to each curve in X is semistable.

On the one hand, this property of $\mathcal{O}_X^{\oplus r}$ is very useful, as we will see in the sequel. On the other hand, the big automorphism group of $\mathcal{O}_X^{\oplus r}$ excludes the existence of a fine moduli space. Consequently, we have to circumvent assumption 3.2.

3.2 The construction

3.6 How to circumvent assumption 3.2? Suppose there exists a smooth curve C in the moduli space $\mathcal{M}$. More generally, we may assume there exists a map $\iota : C \to \mathcal{M}$ from the smooth projective curve C to $\mathcal{M}$. The properness of the moduli space (see [14]) guarantees the existence of the universal bundle $\mathcal{E}_C$ which would be isomorphic to $\iota^*\mathcal{E}$ if the universal bundle $\mathcal{E}$ on $\mathcal{M}$ existed.

3.7 Remark. At first glance, it seems that by restricting our considerations to a curve C in $\mathcal{M}$ we lose some information. Surprisingly, this is not the case if we choose a "big" curve C. This holds because the moduli space of μ-semistable sheaves on a variety Y is embedded into the moduli space of μ-semistable sheaves on D when D is a sufficiently ample divisor on Y. Hence, by taking C to be the normalization of the intersection of $\dim(\mathcal{M}) - 1$ general divisors in a sufficiently ample system $|D|$, we find such a curve C.

3.8 Remark. It is well known, that there exists an etale cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$, such that on $X \times \tilde{\mathcal{M}}$ a (quasi) universal family exists. The obstruction h for the existence of a universal object is a class in the Brauer group $\mathrm{Br}(\mathcal{M})$. Each curve C in $\mathcal{M}$ defines a restriction h_C of h to $\mathrm{Br}(C)$. The existence of the pull back of the "virtual" universal family on $X \times C$ can be deduced from the fact that $h_C = 0$ which follows from Tsen's theorem (see Arcata III.2 in[6]).

3.9 The universal sheaf $\mathcal{E}_C$ and its singularities. We consider the following morphisms $X \xleftarrow{q} X \times C \xrightarrow{p} C$. Suppose that $\mathcal{E}_C$ is the universal sheaf on $X \times C$. We claim that the singular locus Z of the sheaf $\mathcal{E}_C$, i.e., the closed subset where $\mathcal{E}$ is no vector bundle, can be chosen to have a codimension of at least three. To prove this claim we simply remember that on the one hand most points of $\mathcal{M}$ parametrizes vector bundles. Since C is the intersection of ample divisors on $\mathcal{M}$, we may assume that the generic point of C parametrizes a bundle. On the other hand, let $c \in C$ be a point which parametrizes a coherent sheaf $E := \mathcal{E}_c$ which is not a vector bundle. Since the dimension of the singular locus of this torsion free sheaf is at least 2, the claim follows.
Thus, we may assume that $\mathcal{E}_C$ is a vector bundle when we compute the first chern classes of objects $q_!(\mathrm{End}(\mathcal{E}_C) \otimes p^*F)$ for coherent C-sheaves F.
However, if we choose all Chern classes to be zero, then the limit of semistable vector bundles are vector bundles too. Thus, when focusing on our main case, we may assume that Z is empty.

3.10 The line bundle $\mathcal{L}$. Next we define (in 4.1) the line bundle $\mathcal{L}$ to be the generalized Theta bundle on X for the family $\mathcal{E}_C$ of C-vector bundles. As stated above, the possible singularities of $\mathcal{E}_C$ can be ignored for dimensional reasons. Let $q(Z)$ be the image of the singular locus of $\mathcal{E}$ in X. This set has codimension of at least two. As remarked above we may assume Z to be empty, for our construction. If we were constructing generalized Albanese varieties for nontrivial Chern classes, we had to have this set in mind. In section 4 it is shown that (up to this set) the line bundle $\mathcal{L}$ is nef. To be precise we have the

3.11 Theorem. *Let $\iota : Y \to X$ be a curve. Then the degree of the line bundle $\iota^*\mathcal{L}$ is non negative. Furthermore, if the degree of $\iota^*\mathcal{L}$ equals zero, then for all geometric points*

y_1 and y_2 of Y there exists an isomorphism $\mathcal{E}_{y_1} \cong \mathcal{E}_{y_2}$, and all the vector bundles on Y parametrized by C are S-equivalent.

Proof: The proof is basically done in section 4 (see 4.2, 4.4, and 4.5). The only thing we need to check is that at least one point c of the curve C parametrizes a semistable bundle on Y. However, this follows easily from the fact that our moduli space $\mathcal{M}$ contains the point associated to the trivial vector bundle. Thus, the assumptions of corollary 4.4 are fulfilled. □

3.12 The construction of the generalized Albanese morphism. If $\mathcal{L}$ (or some power of it) were base point free, then it would define a morphism $\psi : X \to \mathbb{P}^m$. Let $\varphi : X \to \mathrm{Alb}_r$ be the Stein factorization of ψ, i.e., φ is surjective with connected fibers. By Theorem 3.11 the map φ would meet the requirements of a generalized Albanese variety. If no power of $\mathcal{L}$ is base point free, then the following nef reduction theorem provides us with an Albanese variety up to birational equivalence. In this case we obtain a birational model of the Albanese morphism and variety.

3.13 Theorem. (see Theorem 2.1 in [2], see also [18]) *Let $\mathcal{L}$ be a nef line bundle on a normal projective variety X, then there exists a dominant rational map $X \overset{\mathrm{alb}_r}{\dashrightarrow} \mathrm{Alb}_r(X)$ with connected fibers such that:*

(1) *The line bundle $\mathcal{L}$ is numerically trivial on all compact fibers F of* alb_r *of dimension* $\dim(X) - \dim(\mathrm{Alb}_r(X))$;

(2) *For every general point $x \in X$ and every irreducible curve C passing through x with* $\dim(\mathrm{alb}_r(C)) > 0$, *we have* $\mathcal{L}.C > 0$;

(3) *There exists compact fibers of* alb_r.

Furthermore, the pair $(\mathrm{alb}_r, \mathrm{Alb}_r(X))$ *is uniquely determined up to birational equivalence.*

3.14 Remark. It should at least be noted that when our construction eluded assumption 3.2 assumption 3.3 was automatically made superfluous.

4 The generalized Theta bundle on moduli spaces

4.1 The generalized Theta divisor for vector bundles on a curve. Let $\mathcal{E}$ be a vector bundle of rank r on $S \times C$, where C is smooth projective curve, and S a connected noetherian scheme. The projections are named as indicated: $S \xleftarrow{p} S \times C \xrightarrow{q} C$. We denote by d the degree of $\mathcal{E}_s = q_*(\mathcal{E} \otimes p^* k(s))$ for a geometric point $s \in S$. Let L be a line bundle on C with $\deg(L) = d$. We define the Cartier divisor class Θ_S by

$$\mathcal{O}(\Theta_S) = \det\Big(p_!\big(\mathcal{E} \otimes q^*(\omega_C^{\oplus r} \oplus L^{-2} \oplus \mathcal{O}_C^{\oplus r-1})\big)\Big)^{-1}.$$

Certainly, the reader has noticed that this line bundle $\mathcal{O}(\Theta_S)$ is the $2 \cdot \gcd(r,d)$th power of the generalized Theta bundle defined in [7]. However, it turns out that the above definition is more favorable for our purpose. Furthermore, we are only interested in the positivity of the bundle in question.

By definition the Theta line bundle depends on L. This problem can be solved by fixing a base point either s_0 in S or c_0 in C. In the first case, take L to be the determinant of $\mathcal{E}_{s_0}$, and in the later case set $L := \mathcal{O}_C(c_0)^{\otimes d}$. Since we are only interested in numerical properties of Θ_S, these details do not require any further discussion.

4.2 Proposition. *If S is a smooth projective curve, then the degrees of the generalized Theta divisors Θ_S, and Θ_C coincide.*

Proof: First, let us fix the notation: By g_C and g_S we denote the genera of the curves. Let $s \in S$ and $c \in C$ be geometric points. By F_p and F_q we denote fibers $p^{-1}(s)$ and $q^{-1}(c)$, respectively. As usual, we write $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ for the Chern classes of $\mathcal{E}$. The degrees d_C and d_S of $\mathcal{E}_s$ and $\mathcal{E}_c$ are the intersection numbers $c_1(\mathcal{E}).F_p$ and $c_1(\mathcal{E}).F_q$, respectively. Next we compute the degree of Θ_S. We do so by using the Hirzebruch-Grothendieck-Riemann-Roch theorem. Since we are interested in the degree of the Theta bundle only, we are allowed to write

$$\mathrm{ch}(\omega_C^{\oplus r} \oplus L^{-2} \oplus \mathcal{O}_C^{\oplus r-1}) = 2 \cdot r + (r(2g_C - 2) - 2d_C) \cdot c\,.$$

Applying the definition of Θ_S, the fact that $F_q^2 = 0$, and the Hirzebruch-Grothendieck-Riemann-Roch theorem we obtain:

$$\begin{aligned}
\deg(\Theta_S) &= -\int_S \mathrm{ch}(\Big(p_!\big(\mathcal{E} \otimes q^*(\omega_C^{\oplus r} \oplus L^{-2} \oplus \mathcal{O}_C^{\oplus r-1})\big)\Big) \\
&= -\int_{S\times C} \mathrm{ch}\big(\mathcal{E} \otimes q^*(\omega_C^{\oplus r} \oplus L^{-2} \oplus \mathcal{O}_C^{\oplus r-1})\big) \cdot \mathrm{Td}(p) \\
&= -\int_{S\times C} \mathrm{ch}(\mathcal{E}) \cdot q^*\mathrm{ch}(\omega_C^{\oplus r} \oplus L^{-2} \oplus \mathcal{O}_C^{\oplus r-1}) \cdot \mathrm{Td}(p) \\
&= -\int_{S\times C} \left(r + c_1(\mathcal{E}) + \tfrac{c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})}{2}\right) \cdot \\
&\qquad \cdot \big(2r + (r(2g_C - 2) - 2d_C) \cdot F_q\big) \cdot (1 - (g_C - 1) \cdot F_q) \\
&= -\int_{S\times C} \left(r + c_1(\mathcal{E}) + \tfrac{c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})}{2}\right) \cdot (2r - 2d_C \cdot F_q) \\
&= -\int_{S\times C} (r \cdot (c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})) - 2d_C \cdot F_q \cdot c_1(\mathcal{E})) \\
&= 2d_C d_S - r \cdot \int_{S\times C} (c_1^2(\mathcal{E}) - 2c_2(\mathcal{E}))\ .
\end{aligned}$$

Obviously, this formula is unchanged when interchanging C and S. □

The following result of M. Popa ([17]) will be used in the sequel. It uses the method of J. Le Potier ([16]) to associate sections of the generalized Theta bundle to vector bundles.

4.3 Proposition. ([17] Theorem 5.3) *Let S be a smooth projective curve and E a semistable rank r vector bundle on S. For any integer $k > \frac{r}{4}$, there exists a vector bundle F of rank $2rk$ and $\det(F) \cong \left(\omega_S^{\otimes r} \otimes \det(E)^{-2}\right)^{\otimes k}$ such that $H^*(S, E \otimes F) = 0$.*

4.4 Corollary. *Let S be a smooth projective curve. If there exists a point $c_0 \in C$ such that $\mathcal{E}_{c_0}$ is semistable, then we have $\deg(\Theta_C) \geq 0$. Furthermore, if $\deg(\Theta_C) = 0$ we have $\Theta_C = 0$.*

Proof: If F is a rank $2rk$ vector bundle on S with $\det(F) \cong \left(\omega_S^{\otimes r} \otimes \det(\mathcal{E}_{c_0})^{-2}\right)^{\otimes k}$, then we have $\mathcal{O}_C(k \cdot \Theta_C) = \det\left(q_!(\mathcal{E} \otimes p^*F)\right)^{-1}$. However, if we choose $k > \frac{r}{4}$ and F such that $H^*(\mathcal{E}_{c_0} \otimes F) = 0$, which is possible by the above result, then we obtain an effective divisor in the linear system $|k \cdot \Theta_S|$ not containing the point c_0. □

4.5 Lemma. *Let C and S be smooth projective curves. We suppose there exists a point $c_0 \in C$ such that $\mathcal{E}_{c_0}$ is a semistable vector bundle on S, and the degree of Θ_C is zero. Under these assumptions the vector bundles $\mathcal{E}_P$ and $\mathcal{E}_Q$ are isomorphic for all points P and Q of S.*

Proof: We show that given a point $P \in S$, for almost all points $Q \in S$ we have an isomorphism between $\mathcal{E}_P$ and $\mathcal{E}_Q$. From this statement, the assertion of the Lemma follows immediately. Take a vector bundle F_P on S such that $H^*(S, F_P \otimes \mathcal{E}_{c_0}) = 0$. Such a bundle exists because of Proposition 4.3. Now we consider a nontrivial extension F in $\mathrm{Ext}^1(k(P), F_P)$

$$(\mathcal{S}_P) \qquad 0 \to F_P \to F \stackrel{\pi_P}{\to} k(P) \to 0\,.$$

The scheme $\mathbb{P}(F)$ parametrizes surjections $\pi : F \to k(Q)$ from F to torsion sheaves of length one. The subset of $\mathbb{P}(F)$ where $H^*(S, \ker(\pi) \otimes \mathcal{E}_{c_0}) = 0$ is open and not empty because it contains π_P. Thus, for a general point Q of S there exists a short exact sequence

$$(\mathcal{S}_Q) \qquad 0 \to F_Q \to F \to k(Q) \to 0$$

with $H^*(S, F_Q \otimes \mathcal{E}_{c_0}) = 0$. Applying the functor $R^\bullet q_*(p^*(-) \otimes \mathcal{E})$ to the short exact sequences $(\mathcal{S}_P)$ and $(\mathcal{S}_Q)$, we obtain that $q_*(p^*F \otimes \mathcal{E})$ is isomorphic to $\mathcal{E}_P$ and to $\mathcal{E}_Q$ as well. $\square$

4.6 Remark. If the degree of Θ_C equals zero, then this divisor is rationally equivalent to zero. Unfortunately, for Θ_S we only know that its degree is zero. It can be shown (see [8] Theorem I.4 for the general case or Theorem 3.4 in [10] for a simplified version for the rank two case) that all the semistable vector bundles parametrized by C are S-equivalent. For the family of (not necessarily semistable) vector bundles parametrized by the curve S we find that they are isomorphic.
Regarding a nontrivial extension of line bundles

$$0 \to p^*A \otimes q^*M \to E \to p^*B \otimes q^*N \to 0$$

with $\deg(M) > \deg(N)$ and $\deg(A) = \deg(B)$, we obtain an example where, for the generic choice of A and B, the divisor Θ_S is not even a torsion element.

5 The case of algebraic surfaces

5.1 We consider here the case of a polarized projective algebraic surface (X, H). In this case, we can make a much stronger statement than only theorem 3.13. The generalized Albanese morphism is well understood if X is not of general type (see 5.10). The rest of this section is devoted to the discussion and the proof of the

5.2 Theorem. *Let (X, H) be a polarized projective surface. Then there exists a surjective morphism $\mathrm{alb}_r : X \to \mathrm{Alb}_r(X)$ with connected fibers, and an effective divisor D on X, such that for all morphisms $\iota : C \to X$ of irreducible curves with $\iota(C) \not\subset D$ the following conditions are equivalent.*

(1) $\mathrm{alb}_r(\iota(C))$ *is a point;*

(2) The associated morphism $\mathrm{Hom}(\pi_1(X), \mathrm{U}(r)) \xrightarrow{\iota^*} \mathrm{Hom}(\pi_1(C), \mathrm{U}(r))$ *has finite image;*

(3) For any base scheme S and any rank r vector bundle E on $X \times S$ such that for each s E_s is semistable with numerically trivial Chern classes, the pull back of E to $C \times S$ is a family of S-equivalent vector bundles.

The divisor D of exceptions can be taken to be the empty set when the dimension of $\mathrm{Alb}_r(X)$ *is less than two.*
If $\dim(\mathrm{Alb}_r(X) = 2$, *then the divisor D can be written in the form $D = C_1 + C_2 + \ldots + C_l$, where the C_i are irreducible and form a basis of a proper subspace of the rational Néron-Severi vector space $NS(X) \otimes \mathbb{Q}$. In particular, we have $l < \dim_{\mathbb{Q}}(NS(X) \otimes \mathbb{Q})$.*

5.3 Curves with negative self intersection and their fundamental groups. On the one hand, the Lefschetz hyperplane theorem (see [15] or Theorem 1.1 in [13]) tells us, that any smooth ample divisor $H \overset{\iota}{\hookrightarrow} X$ defines an surjection $\pi_1(H) \overset{\iota_*}{\twoheadrightarrow} \pi_1(X)$.

Loosely, we may interpret a positive self intersection C^2 as implying that the image of $\pi_1(C)$ in $\pi_1(X)$ is big.

On the other hand, any negative divisor C on X is contractible. Unfortunately, the corresponding morphism $\pi_1(C) \to \pi_1(X)$ needs not to be trivial. Here are some examples of this:

Example 1. Let X be the surface obtained by blowing up the product $\mathbb{P}^1 \times C$ in a point, where C is a curve of positive genus. Then X contains a curve $\tilde{C}$ isomorphic to C of self intersection -1. However, the corresponding morphism $\pi_1(\tilde{C}) \to \pi_1(X)$ is a (nontrivial) isomorphism.

Example 2. Consider the projectivization $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(n \cdot P))$ of a sum of two line bundles on a curve C of positive genus. Here P is a geometric point of C and n is a positive integer. It contains a unique curve $\tilde{C}$ of self intersection $-n$ which is isomorphic to C. The resulting map of fundamental groups gives isomorphisms $\pi_1(\tilde{C}) \xrightarrow{\sim} \pi_1(X) \xrightarrow{\sim} \pi_1(C)$.

Example 3. Let C be an algebraic curve of genus $g_C \geq 2$, then the diagonal Δ in the product $C \times C$ has self intersection $2 - 2g_C$. The corresponding morphism of fundamental groups is nothing but the diagonal embedding $\pi_1(C) \to \pi_1(C) \times \pi_1(C)$. (More generally, we could consider embeddings of a curve C in the product $D_1 \times D_2$ of two algebraic curves.)

5.4 Curves C with $C.\mathcal{L} = 0$. However, if C is a rational negative curve, or more generally a tree of rational curves, then the fundamental group $\pi_1(C)$ is trivial. Thus, the generalized Albanese morphisms $X \overset{\mathrm{alb}_r}{\dashrightarrow} \mathrm{Alb}(X)$ should contract these curves. This perception is true, because stable vector bundles with numerically trivial Chern classes can be equipped with a flat metric by the theorem of Uhlenbeck and Yau ([19]). Thus, they correspond to representations of the fundamental group. Consequently $\mathcal{L}.C = 0$, for any tree C of rational curves (contractible or not). The same argument yields that $\mathcal{L}.C = 0$ for any curve C in X such that the corresponding morphisms $\mathrm{Hom}(\pi_1(X), \mathrm{U}(r)) \to \mathrm{Hom}(\pi_1(C), \mathrm{U}(r))$ of fundamental groups has a finite image up to conjugation.

5.5 For an algebraic surface X with the nef line bundle $\mathcal{L}$ (Theorem 3.11) we will construct in the sequel the nef-reduction morphism with respect to $\mathcal{L}$. $\mathcal{L}$ being a nef divisor is the limit of ample divisors (with rational coefficients) which yields $c_1(\mathcal{L})^2 \geq 0$. If for all curves C in X the intersection number $c_1(\mathcal{L}).C = 0$, then $\mathcal{L}$ is numerically equivalent to $\mathcal{O}_X$ and we see that the morphism $\phi_{\mathcal{L}}$ is the structure morphism $X \to \mathrm{Spec}(\mathbb{C})$. Thus, we may assume that there exists a curve C in X with $c_1(\mathcal{L}).C > 0$. This implies $c_1(\mathcal{L}).H > 0$, for all ample divisors H. We will make use of the

5.6 Lemma. *Let (X, H) be a polarized projective surface. As usual, we denote by ρ the Picard number of X which is the rank of the Néron-Severi group* $\mathrm{NS}(X)$. *If $\mathcal{L}$ is a nef line bundle on X which is not numerically trivial, then the number of irreducible curves C satisfying $C^2 < 0$, and $C.\mathcal{L} = 0$ can not exceed $\frac{\rho^2}{4}$.*

Proof: We will divide the proof in steps. First let us fix the notation. Since $\mathcal{L}$ is nontrivial and nef, the intersection $\mathrm{NE}_{\mathcal{L}=0}(X) = \{D \in \mathrm{NE}(X) \mid D.\mathcal{L} = 0\}$ of the effective cone $\mathrm{NE}(X) \subset \mathrm{NS}(X) \otimes \mathbb{Q}$ with the orthogonal hyperplane to $\mathcal{L}$ is contained in the boundary $\partial\mathrm{NE}(X)$ of the effective cone. Thus, let $V = \{C_1, C_2, \ldots\}$ be the (possibly infinite) set of irreducible curves on X satisfying $C_i^2 < 0$, and $C.\mathcal{L} = 0$. By definition these are points in $\mathrm{NE}_{\mathcal{L}=0}(X)$.

Step 1: Nonpositive self intersections. Any nontrivial linear combination $D = \sum_{i=1}^{l} \alpha_i C_i$ with $\alpha_i \geq 0$ is contained in $\mathrm{NE}_{\mathcal{L}=0}(X)$ and satisfies $D.H > 0$. Thus, if D^2 were

positive, then the divisor D would be in the interior of $\mathrm{NE}(X)$ (see Lemma 4.3 in [5]). Since this is a contradiction, we have $D^2 \leq 0$.
Step 2: Linear relations. If we take ρ different curves, say $C_1, \ldots, C_\rho$, of V, then there exists a nontrivial linear relation. This relation may (after permuting indices) be written as $A := \sum_{i=1}^{l} \alpha_i C_i = \sum_{i=l+1}^{\rho-m} \alpha_i C_i =: B$ with all $\alpha_i > 0$, and $m \geq 0$ the number of zero coefficients in the linear relation. We have seen in the first step that $A^2 \leq 0$. On the other hand, the support of A and B consists of different divisors, which implies $A.B \geq 0$. This implies $A.B = A^2 = B^2 = 0$ and eventually the following three statements:
(1) $C_i.C_j = 0$ whenever $i \leq l$, and $l+1 \leq j \leq \rho - m$;
(2) There exists a pair (i,j) of integers with $1 \leq i < j \leq l$ such that $C_i.C_j > 0$;
(3) There exists a pair (i,j) of integers with $l+1 \leq i < j \leq \rho - m$ such that $C_i.C_j > 0$;
Suppose now that $m > 0$. From step 1 we know that $(A+\varepsilon\cdot C_\rho)^2 = 2\varepsilon\sum_{i=1}^{l}\alpha_i C_i.C_\rho + \varepsilon^2 C_\rho^2$ has to be less than or equal to zero for all $\varepsilon \geq 0$. This implies that $C_i.C_\rho = 0$, for all $i = 1, \ldots, l$. An analogous consideration of $(B + \varepsilon \cdot C_\rho)^2$ yields the fourth statement:
(4) If $m > 0$, then $C_i.C_j = 0$ whenever $i \leq \rho - m$, and $\rho - m < j \leq \rho$.
Step 3: Translation to a problem in graph theory. We consider the graph Γ with vertex set V. Two vertices C_i and C_j are adjacent, iff $C_i.C_j > 0$. We have seen in the second step that this graph fulfills the following condition (*):
Any subset $M \subset V$ with exactly ρ elements can be separated into three disjoint subsets $M = M' \cup M'' \cup M'''$ such that:
(1) There exists no adjacency between the three sets M', M'', and M''';
(2) There exists an adjacent pair of vertices in M', as well as in M''.
Step 4: Any graph Γ satisfying (*) has at most $\frac{\rho^2}{4}$ vertices. Let a be the number of connected components of the graph, and b the maximal number of vertices in a connected component. Suppose the inequality $a+b > \rho$ were true, then we could find a subset M of the vertices by taking a vertices of one connected component and $(\rho - a)$ vertices of different connected components. This set M would contradict our condition (*). Consequently, $a + b \leq \rho$ holds. Since the number of vertices is bounded above by $a \cdot b$, the statement of the lemma is true. □

When dealing with curves C which are trivial with respect to $\mathcal{L}$ we may therefore assume that $C^2 \geq 0$. As seen above, this excludes only a finite number of irreducible curves. Let us discuss what can be said about the generalized Albanese morphisms which are dependent on the positivity of $\mathcal{L}^2$.

5.7 The case when $\mathcal{L}^2$ is positive. The Hodge index theorem (IV.2.15 in [1]) asserts that for any effective divisor D the equation $\mathcal{L}.D > 0$ holds, unless $D^2 < 0$. Thus, the set of all irreducible curves C_i with $C_i.\mathcal{L} = 0$ is finite. If the union C of all these curves is projectively contractible, then we take this contraction to be the generalized Albanese morphism. Otherwise we just take the identity of X. Either way theorem 5.2 holds.
If $\mathcal{L}^2 = 0$, then the irregularity q of the surface X must also be taken into consideration.

5.8 The case when $\mathcal{L}^2 = 0$, and $q = 0$. We consider the irreducible curves C with $C^2 \geq 0$, and $C.\mathcal{L} = 0$. Since $\mathcal{L}$ is not numerically trivial, the Hodge index theorem implies that $C^2 = 0$. Again by the Hodge index theorem, we find that these curves C_i can not intersect each other. Thus, they behave like fibers. Indeed, if there are more than $\rho - 1$ of these curves, then we find (as in step 2 of the proof of 5.6) a linear combination $\sum_{i=1}^{l}\alpha_i C_i = \sum_{i=l+1}^{\rho-m}\alpha_i C_i$. After a suitable multiplication we may assume that all the α_i are integers. Since $q = 0$ the line bundle $\mathcal{O}_X(\sum_{i=1}^{l}\alpha_i C_i - \sum_{i=l+1}^{\rho-m}\alpha_i C_i)$ is torsion. Choosing higher multiples of the α_i, we may assume this torsion bundle to be

trivial. We end up with a base point free linear system $|\sum_{i=1}^{l}\alpha_i C_i| = |\sum_{i=l+1}^{\rho-m}\alpha_i C_i)|$. Taking the Stein factorization of the corresponding morphism, we obtain a morphism $\mathrm{alb}_r : X \to \mathrm{Alb}_r(X)$ to a curve. For all fibers F we obviously have $F.\mathcal{L} = 0$. Conversely, let C be an irreducible curve with $C.\mathcal{L} = 0$. If C is not contracted to a point by $\mathrm{alb}_r(X)$, then $C.F > 0$. This implies $(C + nF)^2 > 0$, for $n \gg 0$. However, this was excluded, because of $(C + nF).\mathcal{L} = 0$.
If no linear relation among these curves C_i exists, then we take alb_r just to be the identity, and take the divisor D of theorem 5.2 to be the finite union of all irreducible curves C with $C.\mathcal{L} = 0$.

5.9 The case when $\mathcal{L}^2 = 0$, and $q > 0$. Since q is positive, the Albanese morphism $\mathrm{alb}_1 : X \to \mathrm{Alb}_1(X)$ is nontrivial. The description of its fibers (2.4) tells us that all the curves C with $\mathcal{L}.C = 0$ are fibers of alb_1. Eventually, we obtain that in this case the generalized Albanese morphism is just the Stein factorization of the classical one.
The arguments of 5.8 and 5.9 work whenever we have a linear relation between those curves C which satisfy $C.\mathcal{L} = 0$. This justifies the last statement of Theorem 5.2.

5.10 Surfaces of Kodaira dimension less than 2. Let X be a projective algebraic surface of Kodaira dimension $\kappa(X) \leq 1$. We assume that X is minimal. This is not a restriction as we have seen in 5.4. The next table gives the generalized Albanese morphism for these surfaces following the Enriques-Kodaira classification (see VI. in [1]). The last two rows are perhaps the most interesting ones. They show that the generalized Albanese morphisms may reveal more of the surface than the classical one. We assume in these two rows that the surface X is not a product of two curves.

$\kappa(X)$	class of X	the generalized Albanese morphism
$-\infty$	rational surfaces	$X \to \mathrm{Spec}(\mathbb{C})$
$-\infty$	ruled surfaces $X \to B$ with $g(B) \geq 1$	$X \to B$
0	Enriques surfaces	$X \to \mathrm{Spec}(\mathbb{C})$
0	K3 surfaces	$X \to \mathrm{Spec}(\mathbb{C})$
0	tori	$X \to X$
0	hyperelliptic surfaces	$\mathrm{Alb}_1(X)$ is an elliptic curve, whereas $\mathrm{Alb}_r(X) \cong X$, for $r > 1$, see also §6.
1	properly elliptic surfaces	$\mathrm{Alb}_1(X)$ is an algebraic curve, and $\mathrm{Alb}_r(X) \cong X$, for $r > 1$.

6 An example

6.1 The group G. We consider the group G which as a set is $\mathbb{Z}^4$ with group structure given by $(a, b, c, d)(a', b', c', d') = (a + a', b + b', c + (-1)^a c', d + (-1)^a d')$. The group G has the four generators $g_0 = (1, 0, 0, 0)$, $g_1 = (0, 1, 0, 0)$, $g_2 = (0, 0, 1, 0)$, and $g_3 = (0, 0, 0, 1)$ satisfying the six relations:

$$g_0g_1 = g_1g_0 \quad g_0g_2 = g_2^{-1}g_0 \quad g_0g_3 = g_3^{-1}g_0 \quad g_1g_2 = g_2g_1 \quad g_1g_3 = g_3g_1 \quad g_2g_3 = g_3g_2\,.$$

G may be described as the semidirect product $(\mathbb{Z}^2) \ltimes (\mathbb{Z}^2)$ where the homomorphism $(\mathbb{Z}^2) \to \mathrm{Aut}(\mathbb{Z}^2)$ is given by $(a, b) \mapsto (-1)^a \mathrm{id}_{\mathbb{Z}^2}$.

6.2 Characters of G. It is easy to check that the commutator subgroup $[G, G]$ is generated by g_2^2 and g_3^2. Thus, we obtain $G/[G, G] \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$. Consequently, each group of homomorphism $\tau : G \to \mathbb{C}^*$ must send g_2 and g_3 to ± 1. Thus, in a continuous family τ_t of such morphisms the images of g_2 and g_3 are locally constant. We conclude that there are four families of U(1)-representations of G. One is given by

$$\tau_{t_1,t_2}(g_0) = \exp(t_1 \cdot i) \quad \tau_{t_1,t_2}(g_1) = \exp(t_2 \cdot i) \quad \tau_{t_1,t_2}(g_2) \equiv 1 \equiv \tau_{t_1,t_2}(g_3)$$

with $(t_1, t_2) \in (\mathbb{R}/2\pi\mathbb{Z})^2$. The other three representations of G in U(1) are obtained by changing the image of g_2 or g_3 to -1.

6.3 A family of $\mathrm{SU}(2)$**-representations of** G**.** We consider the following family of representations of $\rho_{t_1,t_2} : G \to \mathrm{SU}(2)$ parametrized by the torus $(\mathbb{R}/2\pi\mathbb{Z})^2$.

$$\rho_{t_1,t_2}(g_0) \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho_{t_1,t_2}(g_2) = \begin{pmatrix} \exp(t_1 \cdot i) & 0 \\ 0 & \exp(-t_1 \cdot i) \end{pmatrix}$$

$$\rho_{t_1,t_2}(g_1) \equiv 1 \qquad \rho_{t_1,t_2}(g_3) = \begin{pmatrix} \exp(t_2 \cdot i) & 0 \\ 0 & \exp(-t_2 \cdot i) \end{pmatrix}$$

It is obvious that the images of ρ_{t_1,t_2} and $\rho_{t'_1,t'_2}$ are not conjugated unless $t'_1 = \pm t_1$ and $t'_2 = \pm t_2$.

6.4 G **is the fundamental group of a hyperelliptic surface.** We consider two elliptic curves $(E_1, 0)$ and $(E_2, 0)$ with a nonzero two-torsion point $e_1 \in E_1$. The fix point free action of $\mathbb{Z}/2\mathbb{Z}$ on $E_1 \times E_2$ generated by $(z_1, z_2) \mapsto (z_1 + e_1, -z_2)$ induces a smooth projective quotient X, and a $2 : 1$ covering $p : E_1 \times E_2 \to X$. This X is a hyperelliptic surface (see [1] V.5). The fundamental group $\pi_1(X)$ is isomorphic to G and the commutative subgroup $2\mathbb{Z} \times \mathbb{Z}^3$ of index two corresponds to the cover p. Denoting the $\mathbb{Z}/2\mathbb{Z}$-quotient of E_1 by e_1 by $\bar{E}_1$, we obtain a morphism $\alpha : X \to \bar{E}_1$. The induced map on the first homology groups

$$\alpha_* : \left[H_1(X, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2\right] \longrightarrow \left[H_1(\bar{E}_1, \mathbb{Z}) \cong \mathbb{Z}^2\right]$$

is an isomorphism between the free parts whereas the torsion part of $H_1(X, \mathbb{Z})$ is mapped to zero. Thus, the morphism α is the Albanese morphism for the surface X.

6.5 Assumption: $\mathrm{Hom}(E_1, E_2) = 0$**.** This assumption will simplify matters. It implies in particular that any line bundle L on the product $E_1 \times E_2$ is of the form $L_1 \boxtimes L_2$. Furthermore, such a line bundle is ample, iff the degrees $\deg(L_1)$ and $\deg(L_2)$ are both positive.

6.6 The generalized Albanese morphism $\mathrm{alb}_2(X)$ **differs from the ordinary Albanese morphism.** Let $\mathcal{L}$ be the line bundle "defining" the generalized Albanese morphism $\mathrm{alb}_2(X)$. As mentioned in 6.5 we can write $p^*\mathcal{L} \cong L_1 \boxtimes L_2$. The almost embedding $\mathrm{U}(1) \to \mathrm{SU}(2)$ defined by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ and the $U(1)$ representation τ of 6.2 imply that $\deg(L_1) > 0$. The representation ρ of 6.3 implies that the degree of L_2 is positive too. Thus, the line bundle $\mathcal{L}$ really defines a morphism $\mathrm{alb}_2 : X \to \mathrm{Alb}_2(X)$ which has to be an isomorphism. On the other hand, the classical Albanese variety has only dimension one.

References

[1] W. Barth; C. Peters; A. Van de Ven, *Compact Complex Surfaces*, Springer Verlag, Berlin, 1984.

[2] Th. Bauer; F. Campana; Th. Eckl; S. Kebekus; Th. Peternell; S. Rams; T. Szemberg; Lorenz Wotzlaw, *A reduction map for nef line bundles*, in I. Bauer; F. Catanese; Y. Kawamata; T. Peternell; Y.-T. Siu (editors), *Complex Geometry, Collection of Papers dedicated to Hans Grauert*, p. 27-36, Springer Verlag, 2002.

[3] F. A. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR Isvestija, **13** (1979), p. 499-555.

[4] F. Campana, *Remarques sur le revêtment universel des variétés Kählériennes compactes*, Bull. Soc. Math. France **122** (1994), p. 255-284.

[5] H. Clemens; J. Kollár; S. Mori, *Higher dimensional complex geometry*, Astérisque **166** (1988).

[6] P. Deligne, *Cohomologie Etale, Séminaire de Géometrie Algébrique du Bois-Marie SGA* $4\frac{1}{2}$, Lecture Notes in Mathematics **569**, Springer-Verlag, Berlin, 1977.

[7] J.-M. Drezet; M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97** (1989), p. 53-94.

[8] G. Faltings, *Stable G-bundles and projective connections*, J. Alg. Geom. **2** (1993), p. 507-568.

[9] P. Griffith; J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1978.

[10] G. Hein, *Duality construction of moduli spaces*, Geometria Dedicata **75** (1999), p. 101-113.

[11] L. Katzarkov, *Factorization theorems for the representations of the fundamental groups of quasiprojective varieties and some applications*, `math.AG/9402012`, preprint 1994.

[12] L. Katzarkov, *On the Shafarevich maps*, Proc. of Symp. in Pure Math. **62.2** (1997) p.173-216.

[13] J. Kollár, *Shafarevich Maps and Automorphic Forms*, Princeton University Press, Princeton NJ, 1995.

[14] S. Langton, *Valuative criteria for families of vector bundles on algebraic varieties*, Ann. Math. **101** (1975), p. 88-110.

[15] S. Lefschetz, *L'analysis situs et la géometrie algébrique*, Gautier Villars, 1924.

[16] J. Le Potier, *Module des fibrés semistables et fonctions thêta*, in M. Maruyama (ed.), *Proc. Symp. Taniguchi Kyoto 1994: Moduli of vector bundles*, Lect. Notes in Pure and Appl. Math. **179** (1996), p. 83-101.

[17] M. Popa, *Dimension estimates for Hilbert schemes and effective base point freeness on moduli spaces of vector bundles*, Duke Math. Journal **107** (2001), p. 469-495.

[18] H. Tsuji, *Numerically trivial fibrations*, `math.AG/0001023`, preprint 2000.

[19] K. Uhlenbeck; S.-T. Yau, *On the existence of of Hermitian Yang-Mills connections on stable vector bundles I & II*, Comm. Pure Appl. Math. **39** (1986) p. 257-293, and **42** (1989) p. 703-707.

Georg Hein, Freie Universität Berlin, Institut für Mathematik II, Arnimallee 3, D-14195 Berlin, Germany, ghein@math.fu-berlin.de