Two constructions of the moduli space of vector bundles on an algebraic curve

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Abstract

We present two constructions to obtain the coarse moduli space of semistable vector bundles on an algebraic curve X. To do so, we survey semistability, Geometric invariant theory, illustrate this theory for the moduli space of elliptic curves, and show the importance of the global sections of the generalized Θ-line bundle associated to semistable vector bundles.

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Key words: moduli space of vector bundles, generalized Theta divisor, algebraic curve

1 Introduction

On November 17, 2003 the author gave a talk Professor Y. Tschinkel's seminar *Geometrische Methoden in der Darstellungstheorie* at the University of Göttingen. These are the slightly extended notes of this lecture. Since the lecture was restricted to sixty minutes, section four of the present presentation was skipped in the lecture.

The plan of this article is to present two constructions to obtain the moduli space of semistable vector bundles on a algebraic curve, and to explain why we have to insert the adjective semistability here. First, we give the setup for these constructions in section 2. Both constructions are based on Grothendiecks Quot schemes which are presented in this section. In the third section, we sketch Mumford's Geometric Invariant Theory (GIT). Then, we illustrate this approach for the moduli space of elliptic curves in section 4. This section is included because, in this case, we can very precisely describe the invariant forms and the invariant subring. In section 5, we apply the GIT to our moduli problem whereas in section 6, we present an alternative approach to the construction of this moduli space. However, we also point out that this constructin can be understood in the language of the GIT, provided we understand the invariant sections.

There is little that is original in what I present here. Perhaps the presentation of topics in the last section is novel. However, the author is responsible for all mistakes.

The author wishes to express his gratitude for being invited to the lecture and that he appreciated the atmosphere during his short stay in Göttingen.

2 The setup

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k. Furthermore, we fix a globally generated ample divisor $\mathcal{O}_X(1)$. We want to study vector bundles on X. As usual for a sheaf F, we write $F(m)$ for the twist $F \otimes \mathcal{O}_X(1)^{\otimes m}$. There are two discrete invariants, i.e. invariants which do not vary in flat familes over connected schemes, namely the rank r and the degree d of the vector bundle. If the rank is one, then we obtain the Picard variety $\mathrm{Pic}^d(X)$ of all isomorphism classes of line bundles with fixed degree d, they form a $\mathrm{Pic}^0(X)$-torsor. The smooth variety $\mathrm{Pic}^d(X)$ is a paradigm for a moduli space. In the following we investigate the case when $r \geq 2$. Furthermore, we assume that the genus g of X is at least two.
We want to equip the following set with the structure of a projective variety

$$\mathcal{M}_X(r,d) = \left\{ \begin{array}{c} \text{isomorphism classes of vector bundles } E \text{ on } X \text{ with} \\ \mathrm{rk}(E) = r \ \deg(E) = d \end{array} \right\}.$$

When considering a vector bundle E as a point in this space (so far it is only a set), we write $[E]$ for the corresponding point in $\mathcal{M}_X(r,d)$. It turns out that we have to modify the definition of $\mathcal{M}_X(r,d)$ to achieve this goal. For a fixed integer a we set

$$\mathcal{M}_X^a(r,d) = \{[E] \in \mathcal{M}_X(r,d) \mid H^1(E(a-1)) = 0\}.$$

Since $\mathcal{M}_X(r,d) = \cup_{a\in\mathbb{Z}} \mathcal{M}_X^a(r,d)$, we content ourselves with the construction of $\mathcal{M}_X^a(r,d)$. The theory of the Castelnuovo-Mumford regularity (see [8]) implies that for $[E] \in \mathcal{M}_X^a(r,d)$, the twisted vector bundle $E(a)$ is globally generated and has $h^0(E(a)) = \chi(E(a))$. Thus, the number $M := h^0(E(a))$ does not depend on the choice of $[E] \in \mathcal{M}_X^a(r,d)$. Fixing an isomorphism $H^0(E) \cong k^{\oplus M}$, we obtain a surjection $F \twoheadrightarrow E$ for $F = \mathcal{O}_X(-a)^{\oplus M}$.
We have seen that Grothendieck's Quot scheme $\mathrm{Quot} := \mathrm{Quot}_{X/F}^{r,d}$ of quotients of F with rank r and of degree d parametrizes all vector bundles of $\mathcal{M}_X^a(r,d)$. Of course this scheme is much bigger than the space we want, and it parametrizes sheaves which have nontrivial torsion. However, for the subset

$$\mathrm{Quot}(E) := \{[F \to G] \in \mathrm{Quot} \mid \text{there exists an isomorphism } G \stackrel{\alpha}{\to} E\}$$

the following holds:

- $\mathrm{Quot}(E)$ is irreducible, and
- The quotients $[\pi : F \to G]$ in $\mathrm{Quot}(E)$ which induce an isomorphism $H^0(\pi) : H^0(F(a)) \to H^0(G(a))$ form a nonempty open subset in $\mathrm{Quot}(E)$.

There exists a natural $\mathrm{SL}(M)$ action on Quot and, set theoretically, our space $\mathcal{M}_X^a(r,d)$ can be identified with some subset of the quotient of this group action. Thus, we end up with the task of defining a scheme structure on the quotient space of a group action. There exists a general theory for doing so which we briefly sketch in the next section.

3 Geometric invariant theory

3.1 Group actions on schemes and quotients. The general problem of the Geometric invariant theory is: Given a group G and a scheme X together with a group action $G \times X \to X$. Does there exist a "space of the orbits" Y in the category of schemes. Further questions are:

- How far is X from being a G-bundle over Y?
- Which geometric properties of Y can we deduce from X (and G)?
- How can we describe sheaves on Y by sheaves on X?

3.2 Group actions on affine schemes. If $X = \mathrm{Spec}(A)$ is affine, then suppose that there exists a G-invariant map $f : X \to \mathrm{Spec}(B)$. The corresponding ring morphism $\phi : B \to A$ must have its image in the subring A^G of the G-invariant elements of A. This motivates the following: We set $X//G := \mathrm{Spec}(A^G)$. The inclusion $A^G \to A$ induces a dominant morphism $X \to X//G$. The above consideration shows that every G-invariant morphism $X \to \mathrm{Spec}(C)$ factors through $X//G$. Thus, $X//G$ is a categorial quotient.

3.3 A bad group action. Suppose that we consider the group action $\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2$ given by multiplication, i.e. $(\lambda, (x, y)) \mapsto (\lambda x, \lambda y)$. The set of orbits is $\mathbb{P}^1$ and the point $(0, 0)$ which lies in the closure of every orbit. Consequently, there are no $\mathbb{C}^*$ invariant functions on $\mathbb{C}[x, y]$ which can seperate different orbits. Indeed, a simple calculation shows that $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}$. Thus, $\mathbb{C}^2//\mathbb{C}^*$ is just $\mathrm{Spec}(\mathbb{C})$. This example shows how far $X//G$ can be from an orbit space. It also explains why we use the notation $X//G$ instead of X/G.

3.4 Linearization of group actions on quasiprojective schemes. For X quasiprojective with an ample line bundle $\mathcal{O}_X(1)$, we need a linearization of the action. First we require that for all $g \in G$ there exists an isomorphism $g^*\mathcal{O}_X(1) \cong \mathcal{O}_X(1)$. In the case that X is projective and G is affine, this requirement is not too hard, because the map from $G \to \mathrm{Pic}(X)$ defined by $g \mapsto g^*\mathcal{O}_X(1) \otimes \mathcal{O}_X(-1)$ is locally constant. Thus, by replacing $\mathcal{O}_X(1)$ with the product $g_1^*\mathcal{O}_X(1) \otimes \ldots \otimes g_k^*\mathcal{O}_X(1)$i, where each g_i comes from a different connected component of G, we obtain such a G invariant polarization. Suppose that $\mathcal{O}_X(1)$ is G invariant. We consider the two morphisms act (group action) and pr_X (projection)

$$G \times X \underset{\mathrm{pr}_X}{\overset{\mathrm{act}}{\rightrightarrows}} X$$

from $G \times X$ to X. An isomorphism $\beta : \mathrm{act}^*\mathcal{O}_X(1) \to \mathrm{pr}_X^*\mathcal{O}_X(1)$ is called a linearization of the group action if it fullfils a cocycle condition. The existence of the isomorphism β implies that $\mathrm{pr}_G(\mathrm{act}^*\mathcal{O}_X(1) \otimes \mathrm{pr}_X^*\mathcal{O}_X(-1))$ is the trivial line bundle on G. We may interpret β as a family of isomorphisms $\beta_g : g^*\mathcal{O}_X(1) \cong \mathcal{O}_X(1)$ for all $g \in G$. The cocycle condition is equivalent to $\beta_{g\circ h} = h^*(\beta_g) \circ \beta_h$ for all $g, h \in G$. From now on, we suppose that a linearization of the group action exists. Indeed, in most examples we consider the linearization is used to define the group action.

3.5 The GIT quotient of a quasi projective scheme. Let $R = \oplus_{k\geq 0} R_k$ with $R_k = H^0(\mathcal{O}_X(k))$ the graded ring of the pair $(X, \mathcal{O}_X(1))$. The linearization defines an action of G on R_k. Thus, we obtain the subring $R^G \subset R$ which defines a rational map

$$X = \mathrm{Proj}(R) \dashrightarrow \mathrm{Proj}(R^G) =: X//G \ .$$

If the group G is reductive, then the ring R^G is noetherian. We see that by definition $X//G$ is projective. However, we have only a rational morphism from X to $X//G$, because the G invariant homogeneous functions may have base points. These base points are called unstable points of X with respect to G and the chosen polarization. In contrast, we define semistable and stable points as follows:

semistable $x \in X$ is semistable iff there exists a G invariant homogeneous k-form which does not vanish on x. The open set of all semistable points is denoted by X^{ss}.

stable $x \in X$ is called stable iff x is semistable and its orbit is closed in X^{ss} and its stablizer is finite. The open set of all stable points is denoted by X^s.

Thus, we obtain a morphism $X^{ss} \to X//G$. This morphism becomes a fibration of orbits when restricted to X^s.

4 An example: The GIT moduli space of elliptic curves

4.1 Parametrizing elliptic curves. Let (E, P) be an elliptic curve with a geometric point $P \in E$ which is the zero point for the group scheme E. The linear system $|2P|$ is globally generated and of dimension one. Thus, we get a morphism $\psi : E \to \mathbb{P}^1$. By the Hurwitz formula, the set C of the critical values of ψ has exactly four elements $C = \{x_1, x_2, x_3, x_4\}$. On the other hand, we can reconstruct E from the set C as follows: Consider C as a subscheme of $\mathbb{P}^1$. This way we obtain an embedding α_C from the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-4) \stackrel{\alpha_C}{\to} \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_C \to 0\,.$$

Now we fix an isomorphism $\beta : \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}(-4)$ and define an $\mathcal{O}_{\mathbb{P}^1}$-algebra structure on $A_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ locally by $(f, g)\cdot(f', g') = (ff' + \alpha_C(\beta(g \otimes g')), fg' + f'g)$. We now observe that $E \cong \mathrm{Spec}(A_C)$. Two elliptic curves E and E' are isomorphic, iff the corresponding divisors C and C' are mapped to each other by an automorphism of $\mathbb{P}^1$.

4.2 The GIT setting. If V is a vector space of dimension two, then we let $\mathbb{P}^1$ be the space $\mathbb{P}(V^\vee)$. This might seem a little confusing. However, it will simplify matters. Since a divisor C on $\mathbb{P}^1$ of degree four is given by a homogeneous 4-form, up to multiplication, we see that these divisors are parametrized by $\mathbb{P}(\mathrm{Sym}^4(V))$. The group $\mathrm{SL}(V)$ acts on $\mathbb{P}^4 = \mathbb{P}(\mathrm{Sym}^4(V))$. The quotient $X//G$ is the GIT moduli space of elliptic curves.
Let us apply the geometric invariant theory to the group action

$$\mathrm{SL}(V) \times \mathbb{P}(\mathrm{Sym}^4(V)) \to \mathbb{P}(\mathrm{Sym}^4(V))\,.$$

The graded ring R of the pair $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$ is given by

$$R = \bigoplus_{n \geq 0} R_n \quad \text{with} \quad R_n = \mathrm{Sym}^n(\mathrm{Sym}^4(V))\,.$$

4.3 The graded ring $R^{\mathrm{SL}(V)}$. The representations of $\mathrm{SL}(V)$ are well understood, see §11 in [4] for an elementary introduction which includes all we need to know for our purpose. The finite dimensional representation $\mathrm{Sym}^n(\mathrm{Sym}^4(V))$ decomposes into a direct sum of irreducible representations:

$$\mathrm{Sym}^n(\mathrm{Sym}^4(V)) = \bigoplus_{k=0}^{2n} \mathrm{Sym}^{2k}(V)^{\oplus a_{n,k}}\,.$$

Obviously the $\mathrm{SL}(V)$ invariant vectors in R_n form the direct summand $\mathrm{Sym}^0(V)^{\oplus a_{n,0}}$. Computing weights we obtain the following formula for $a_{n,0}$:

$$a_{n,0} = \#\{(k,l) \in \mathbb{Z}^2 \text{ with } l \geq 0\,, k \geq 0\,, \text{ and } 3l + 2k = n\,\}\,.$$

Indeed, the representation $\mathrm{Sym}^{2k}(V)$ has weights $\{-2k, -2(k-1), -2(k-2), \ldots, 2(k-1), 2k\}$, all with multiplicity one. Hence, the space W_{2k} of weight $2k$ vectors in $\mathrm{Sym}^n(\mathrm{Sym}^4(V))$ has dimension $\sum_{l=k}^{2n} a_{n,l}$. On the other hand, using the fact that the weights of $\mathrm{Sym}^4(V)$ are $\{-4, -2, 0, 2, 4\}$, we obtain that the space W_{2k} is of dimension w_{2k} with

$$w_{2k} = \# \left\{ (a_{-2}, a_{-1}, a_0, a_1, a_2) \in \mathbb{Z}^5 \text{ with } \begin{array}{c} a_k \geq 0 \text{ for } k = -2, \ldots, 2 \\ a_{-2} + a_{-1} + a_0 + a_1 + a_2 = n \\ -2a_{-2} - a_{-1} + a_1 + 2a_2 = k \end{array} \right\} .$$

This reduces the computation of $a_{n,0} = w_0 - w_2$ to a purely combinatorial problem. The author, not a specialist in this type of problem, was somehow able to solve it but does not want to bother the reader with his proof. Please believe the above formula.
The following table shows the dimensions $a_{n,0}$ for the first integers n.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$a_{n,0}$	1	0	1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4

4.4 Using the relation $a_{n+6,0} = 1 + a_{n,0}$, we can easily compute the Poincaré series of the graded ring $R^{\mathrm{SL}(V)}$.

$$H(R^{\mathrm{SL}(V)}, t) = \sum_{n \geq 0} a_{n,0} t^n = \frac{1}{(1-t)^2(1+t)(1+t+t^2)} = \frac{1}{(1-t^2)(1-t^3)}$$

This is the Poincaré series of a ring with two generators, one in weight two, and one in weight three. Next we will see that this predicted structure is indeed the structure of $R^{\mathrm{SL}(V)}$.

4.5 An algebra basis of $R^{\mathrm{SL}(V)}$. Let $\{e_0, e_1\}$ be a basis of V. Then we take the associated basis $\{X_0, \ldots X_4\}$ of $\mathrm{Sym}^4(V)$ with $X_i = e_0^i e_1^{4-i}$. It is easy to check that the two homogeneous polynomials

$$\begin{aligned} g_2 &= 12X_0X_4 - 3X_1X_3 + X_2^2 \\ g_3 &= 72X_0X_2X_4 - 27X_0X_3^2 - 27X_1^2X_4 + 9X_1X_2X_3 - 2X_2^3 \end{aligned}$$

are $\mathrm{SL}(V)$ invariant. Moreover, there exits no algebraic relation between these two polynomials. Thus, we obtain an inclusion of $\psi : \mathbb{C}[g_2, g_3] \to R^{\mathrm{SL}(V)}$. Since the polynomial $g_2^m g_3^n$ is homogenous of degree $2m + 3n$, we obtain that

$$\dim(\mathrm{im}(\psi) \cap R_k) = \#\{(m, n) \in \mathbb{Z}^2 \mid m \geq 0\,,\ n \geq 0\,,\ 2m + 3n = k\}\,.$$

We conclude that ψ is an isomorphism.

4.6 Stability of orbits. Now we want to investigate whether an orbit is (semi)stable or not. If $C = \sum_i a_i P_i$ is an effective divisor of degree d on $\mathbb{P}^1$, then we call the decreasing sequence of multiplicities $(a_1, \ldots, a_k)$ the type of C. This gives a stratification on the space of all effective divisors. The number of strata is the number of partitions of the integer d. If $d = 4$, then we have only five strata. In the next table, we list these five types. For each type we give the dimension of the stratum, an example divisor, the corresponding point in $\mathbb{P}^4$ (which is nothing as the coefficients of a polynomial vanishing at those four points) and the values of g_2 and g_3 for this point (here we replaced $(X_0 : X_1 : X_2 : X_3 : X_4)$ by $(X_0, X_1, X_2, X_3, X_4)$).

type	dimension of stratum	example	$X = (X_0 : X_1 : X_2 : X_3 : X_4)$	$g_2(X)$	$g_3(X)$
(1,1,1,1)	4	$(0; 1; \lambda; \infty)$	$(0 : 1 : -1 - \lambda : \lambda : 0)$	$\lambda^2 - \lambda + 1$	$2\lambda^3 - 3\lambda^2 - 3\lambda + 2$
(2,1,1)	3	$(0; 0; 1; \infty)$	$(0 : 1 : -1 : 1 : 0)$	1	2
(2,2)	2	$(0; 0; \infty; \infty)$	$(0 : 0 : 1 : 0 : 0)$	1	-2
(3,1)	2	$(0, 0, 0, \infty)$	$(0 : 1 : 0 : 0 : 0)$	0	0
(4)	1	$(0, 0, 0, 0)$	$(1 : 0 : 0 : 0 : 0)$	0	0

4.7 Meet the j-invariant. The ratio $(g_2^3 : g_3^2)$ is consequently an invariant of a divisor of degree four modulo the $\mathrm{SL}(V)$ action. Those divisors where g_2 and g_3 vanish form the unstable locus. The function $j := \frac{2^8 3^2 g_2^3}{4g_2^3 - g_3^2}$ is called the j-invariant of the elliptic curve corresponding to that divisor. The denominator is different from zero iff the curve is smooth.
We can now conclude the study of the orbits:

type	stability	reasons	geometric meaning
(1,1,1,1)	stable	the j-invariant separates the orbit from the other orbits	smooth elliptic curve
(2,1,1)	semistable, but not stable	$(g_2^3 : g_3^2)$ is defined, but cannot separate this orbit from the type (2,2) orbit	irreducible nodal rational curve
(2,2)	semistable, but not stable	$(g_2^3 : g_3^2)$ is defined, but the stabilzer is of dimension one	two rational curves identified in two different points
(3,1)	unstable	$(g_2^3 : g_3^2)$ is not defined	irreducible rational curve with a cusp
(4)	unstable	$(g_2^3 : g_3^2)$ is not defined	two rational curves identified in a point with tangent direction

4.8 Conclusion. Thus, the GIT-moduli space of elliptic curves is (when constructed in this way) the weighted projective space $\mathbb{P}[2, 3]$. Ignoring this structure by passing to the subring generated by g_2^3 and g_3^2, some people just identify it with $\mathbb{P}^1$ and say "The moduli space of elliptic curves is the affine line, and it is compactified by the point at infinity which corresponds to the nodal cubic curve."

5 The GIT construction of the moduli space of vector bundles

5.1 Stable and semistable points in the Quot scheme Quot**.** Now we return to the situation in section 2. We have to decide whether a quotient $[F \xrightarrow{\pi} E] \in \mathrm{Quot}$ is stable (resp. semistable) for the $SL(M)$ action. For a coherent sheaf G, we call the quotient $\frac{\deg(G)}{\mathrm{rk}(G)}$ the slope of G and denote it by $\mu(G)$. Using the Hilbert-Mumford criterion, we find (see [6] or [9]) that $[F \xrightarrow{\pi} E]$ is semistable if the following two properties hold:

(i) The induced linear map $H^0(\pi) : H^0(F(a)) \to H^0(E(a))$ is an isomorphism.

(ii) For all quotients $E \to G$ the inequality $\mu(E) \leq \mu(G)$ holds.

The second property is independent of α. Hence, it is a property of E itself, which is called semistability. If for all quotients $E \to G$ the inequality $\mu(E) < \mu(G)$ holds, then E is called stable. Having the properties of $\mathrm{Quot}(E)$ in mind (see the end of section 2), we obtain that $\mathcal{M}_X^a(r,d) = \mathrm{Quot}(E)//\mathrm{SL}(M)$ is the moduli space of semistable bundles on the algebraic curve X. Next we see that there is no need to fix the number a.

5.2 Semistability and boundedness. Next we show that a semistable sheaf E is contained in $\mathcal{M}_X^a(r,d)$ for all $a > 2g-1-\mu(E)$. Indeed, suppose that this inequality holds and $[E] \notin \mathcal{M}_X^a$. Hence, $h^1(E(a-1)) > 0$ and, by Serre duality, there exists a nontrivial morphism $\alpha : E \to \omega_X(1-a)$. If E is semistable, then we conclude that $\mu(E) \leq \mu(\omega_X(1-a)) = 2g-1-a$. Thus, a contradiction.
Hence, the parameter a of section 2 can be ignored after all. We just set $a := 2g - \frac{d}{r}$ and we are done. However, all those sheaves in $\mathcal{M}_X^{a+k}(r,d) \setminus \mathcal{M}_X^a(r,d)$ cannot be parametrized by the GIT moduli space, because they are not semistable.
On the one hand, semistability gives rise to a noetherian moduli scheme. On the other hand, we do not obtain the moduli space of all vector bundles, because we are to restricted to the semistable ones.

5.3 Properties of the moduli space $\mathcal{M}_X(r,d)$.

1. $\mathcal{M}_X(r,d)$ is a normal reduced projective variety of dimension $r^2(g-1)+1$.
2. The singular locus of $\mathcal{M}_X(r,d)$ is the set of all semistable bundles which are not stable. In particular, $\mathcal{M}_X(r,d)$ is smooth, iff r and d are coprime. (There exists a single exception to this rule: When $g=2$, $r=2$, and d is even then $\mathcal{M}_X(r,d)$ is smooth.)
3. The map induced by the assignment $E \mapsto \det(E)$ gives rise to a morphism $\det : \mathcal{M}_X(r,d) \to \mathrm{Pic}^d(X)$. The fiber $\det^{-1}(L)$ is denoted by $\mathcal{M}_X(r,L)$.
4. For L_1 and L_2 in $\mathrm{Pic}^d(X)$ there exists (not canonically) an isomorphism $\mathcal{M}_X(r,L_1) \cong \mathcal{M}_X(r,L_2)$. Thus, $\mathcal{M}_X(r,d)$ is a $\mathcal{M}_X(r,L)$ fiber bundle over $\mathrm{Pic}^d(X)$. Therefore, to understand $\mathcal{M}_X(r,d)$ it is almost sufficient to study $\mathcal{M}_X(r,L)$.
5. The Picard group $\mathrm{Pic}(\mathcal{M}_X(r,L))$ is infinite cyclic. The ample generator is called the generalized Θ line bundle.

5.4 The S-equivalence of semistable bundles. It remains to discuss the semistable vector bundles which are not stable. These vector bundles are called properly semistable bundles. The GIT states that two properly semistable points are identified if the closure of their orbits intersect in the semistable locus. If E is an extension of two semistable sheaves E' and E'' of the same slope, then the closure of the orbit also contains the trivial extension $E' \oplus E''$. The smallest equivalence relation which relates E to this direct sum is called the S-equivalence. Note that any semistable vector bundle is S-equivalent to a direct sum of stable vector bundles of the same slope. The moduli space $\mathcal{M}_X(r,d)$ parametrizes S-equivalence classes of semistable vector bundles.

6 Construction of the moduli space without GIT

6.1 We propose an alternative construction of the moduli space which does not use the GIT. However, we are in some sense reversing the chronological order by doing so. This construction can be found in Faltings' article [3]. See also the author's article [7] for a

slightly different presentation and an extension of this construction to the moduli spaces of vector bundles on algebraic surfaces. This construction applies to every moduli space which is constructed using GIT as we point out next:
Let $G \times Y \to Y$ be a group action on the scheme Y and let $Y \overset{\pi}{\dashrightarrow} Z := Y//G$ be the GIT quotient. We suppose that G has connected orbits. Let L be a base point free and ample line bundle on Z. We assume that π^*L extends to a line bundle M on Y. Suppose that the sections $\{s_i\}_{i=0,\dots,N}$ generated L and their pull backs $\{t_i\}_{i=0,\dots,N}$ generate M on the semistable locus. Then we obtain a rational morphism $Y \overset{(t_0:\cdots:t_N)}{\dashrightarrow} \mathbb{P}^N$. Since we did not assume that L is very ample, Z is not the image of this morphism but its Stein factorization.
Thus, once we understand the GIT quotient Z and an ample line bundle with its global sections, we can give a further construction of the moduli space. The one presented here for the moduli space of vector bundles on an algebraic curve is based on the global sections of the generalized Θ-line bundle which correspond to vector bundles. These sections span a linear system in the space of the G invariant sections.

6.2 Six equivalent formulations for semistability

A coherent sheaf E on an irreducible curve X is semistable if one of the following equivalent conditions is satisfied:

(i) E is torsion free, and for all $F \subset E$, we have $\mu(F) \leq \mu(E)$.

(ii) For all quotients $E \to G$ of positive rank, we have $\mu(E) \leq \mu(G)$.

(iii) For all $F \subset E$, we have $\chi(F) \cdot \mathrm{rk}(E) \leq \chi(E) \cdot \mathrm{rk}(F)$.

(iv) For all quotients $E \to G$, the inequality $\chi(E) \cdot \mathrm{rk}(G) \leq \chi(G) \cdot \mathrm{rk}(E)$ holds.

(v) There exists a nontrivial coherent sheaf F on X such that $H^*(X, F \otimes E) = 0$.

(vi) There exists a nontrivial coherent sheaf F of rank $r^3(g_X - 1)$ of X such that all cohomology groups of $F \otimes E$ vanish.[1]

6.3 Sections of the generalized Θ line bundle associated to vector bundles. We consider here the universal family on the Quot scheme Quot of section 2. This means we consider the morphisms

$$\begin{array}{ccc} \mathrm{Quot} \times X & \xrightarrow{p} & X \\ \downarrow{\scriptstyle q} & & \\ \mathrm{Quot} & & \end{array}$$

and the short exact sequence $0 \to \ker(\pi) \to p^*F \xrightarrow{\pi} \mathcal{G} \to 0$ induced by the surjection π. Let G be a rank $r^3(g-1)$ vector bundle on X with $\mu(G) = (g-1) - \frac{d}{r}$ which satisfies $h^0(G(-a)) = 0$. We obtain a long exact sequence on Quot:

$$0 \to q_*(\ker(\pi) \otimes p^*G) \to q_*(p^*F \otimes p^*G) \to q_*(\mathcal{G} \otimes p^*G) \to$$

$$\to R^1q_*(\ker(\pi) \otimes p^*G) \xrightarrow{\gamma} R^1q_*(p^*F \otimes p^*G) \to R^1q_*(\mathcal{G} \otimes p^*G) \to 0\,.$$

[1] The rank condition is for curves of genus at least two. It can be replaced by rank one in the case of a rational curve, and by r in case of a curve of genus one.

The condition $h^0(G(-a)) = 0$ implies that $q_*(p^*F \otimes p^*G) = 0$. Thus, we can compute the ranks of the two vector bundles $A_1 := R^1q_*(\ker(\pi) \otimes p^*G)$ and $A_0 := R^1q_*(p^*F \otimes p^*G)$ by the Riemann Roch theorem to obtain that γ is a morphism of two vector bundles of the same rank R. Thus, we obtain a section $\theta_G := \det(\gamma)$ in the line bundle $\mathcal{O}_{\mathrm{Quot}}(\Theta_G) := \Lambda^R A_0 \otimes (\Lambda^R A_1)^{-1}$. It is not to difficult to see that the line bundles Θ_G and $\Theta_{G'}$ are isomorphic for $\det(G) \cong \det(G')$. Fixing a determinant, i.e. considering only bundles G_i which coincide with G in the Grothendieck group $K_0(X)$ and fixing isomorphisms $\mathcal{O}(\Theta_{G_i}) \cong \mathcal{O}(\Theta_G)$, we obtain for each of the vector bundles G_i a section in $\mathcal{O}(\Theta_G)$. From part (vi) of 6.2 it follows that the base locus of these sections is exactly the unstable locus, i.e. the set of all sheaves parametrized by Quot which are not semistable.

6.4 Semipositivity and S-equivalence. Having the above sections and the information about the moduli space, we can obtain $\mathcal{M}_X(r,d)$ by the construction pointed out in 6.1. However, the ampleness of the Θ-line bundle can be read off from the following result of Faltings.

Theorem. (I.4 in [3]) *Let $\mathcal{E}$ be family of vector bundles on $X \times C$ where C is a smooth projective curve. Suppose that for the generic point η of C the X bundle $\mathcal{E}_\eta$ is semistable. Then the following holds:*
(i) The Θ divisor Θ_C on the curve C is effective.
(ii) If $\Theta_C = 0$, then all X-vector bundles parametrized by C are S-equivalent.

For an easy proof in the rank two case see 3.4 in the author's article [7]. This theorem guarantees that the morphism constructed using the sections of $\mathcal{O}(\Theta_G)$ can distinguish different S-equivalence classes. (Possibly after Stein factorization.) Here we again use the fact, pointed out in section 2, that the subscheme $\mathrm{Quot}(E)$ of the Quot scheme Quot is connected.

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